Letters to the Editor

Perturbation Theory of Relativistic Eigenvalue Problem

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In the recent work on the mass-corrections to the fine structure of hydrogen-like atoms, Salpeter has presented the perturbation theory of the relativistic eigenvalue problem for two-particle system.1) The formulation is essentially based on the facts: there is no external field and thus we have only to do with the coordinates, the motion of the center of gravity being completely separated. Moreover, the unperturbed interaction between particles is assumed to be instantaneous. This reduces rigorously the equation to the one involving only equal time variables for the two particles which corresponds to the Breit equation. His method is, accordingly, not applicable to the cases where there is an external field or the unperturbed interaction is noninstantaneous. We shall offer in this note a primitive but more general perturbation theory including Salpeter's as a special case.

We consider two electrons in an external field which is assumed to be independent of time.²⁾ Such a system is described by the relativistic equation

$$\left[\left(i \gamma (P - e A) + M \right)_a \left(i \gamma (P - e A) + M \right)_b - \bar{I} \right] \psi = 0 ,$$
 (1)

where P_{μ} stands for $1/i \cdot \partial/\partial x_{\mu}$, A is the external field, M the mass operator, \bar{I} the interaction kernel. a and b label the respective electrons. We replace the mass operator M by the electron mass m and consider the remaining terms as being absorbed into $\bar{I}^{(i)}$. We also disregard the effects of pair creation by the external field. Defining the total energy and the relative energy by

$$(P_{ao} + P_{bo}) = E \text{ and } 1/2 \cdot (P_{ao} - P_{bo}) = \varepsilon$$
 respectively, (1) is rewritten as

$$(F-I)\psi=0, \qquad (1')$$

where $I = \beta^{\alpha} \beta^{b} \overline{I}$ and

$$\begin{split} F(E, \mathbf{\varepsilon}) &= [1/2 \cdot E - H_a(\mathbf{P}_a) + \mathbf{\varepsilon}] \\ &\times [1/2 \cdot E - H_b(\mathbf{P}_b) - \mathbf{\varepsilon}] , \\ H_a(\mathbf{P}_a) &= [\mathbf{a} \cdot (\mathbf{P}_a - e\mathbf{A}) + eA_0 + m\beta^a] , \\ H_b(\mathbf{P}_b) &= [\mathbf{a} \cdot b(\mathbf{P}_b - e\mathbf{A}) + eA_0 + m\beta^b] . \end{split}$$

Now, we divide the total energy E into the unperturbed energy E_0 and the perturbed $\Delta E = E_1 + E_2 + \cdots$, and expand, correspondingly, F as

$$F=F_0+(\Delta E)F'+(\Delta E)^2F''$$

in which $F_0=F(E_0,\,\epsilon)$, $F'=1/2\cdot(E_0-H_\alpha(P_\alpha)-H_b(P_b))$ and F''=1/4. The interaction kernel I and the wave function ψ are also divided into the unperturbed part $I_0,\,\psi_0,\,$ and the perturbed $\Delta I=I_1+I_2+\cdots$. Substituting these expansions into (1') and picking up the terms of the same order, we can determine the perturbation energy and the wave function successively. First, as the zeroth order terms, we have the unperturbed equation

$$(F_0 - I_0) \psi_0 = 0. (3)$$

The instantaneity of I_0 is not necessary, although one could hardly solve (3) without it.

We proceed next supposing that the solution ψ_0 and the Green's function G_0 of the equation (3) have been obtained by some method. The first order terms of (1') is the equation

$$(F_0 - I_0)\psi_1 + E_1 F' \psi_0 - I_1 \psi_0 = 0, \qquad (4)$$

which determines E_1 and ψ_1 : From the inner product of (4) with ψ_0 and the hermite character of I_0 , we have the first order perturbation energy

$$E_1 = (\psi_0, I_1 \psi_0) / (\psi_0, F' \psi_0). \tag{5}$$

The denominator and the numerator of (5) have a common divergent factor, i.e., $\int d^4X$ in case of no external fields or $\int dX_0$ in case of time-independent external fields, X_μ being the coordinates of the center of gravity. In what follows we suppose such a factor to be dropped. We also obtain from (4)

$$\psi_1 = G_0 I_1 \psi_0 - E_1 G_0 F' \psi_0. \tag{6}$$

In the same way, the second order terms of (1') gives

$$\begin{split} E_2 = 1/(\psi_0, F'\psi_0) \cdot \{(\psi_0, I_2 \psi_0) + (\psi_0, I_1 G_0 I_1 \psi_0) \\ - E_1(\psi_0, I_1 G_0 F'\psi_0) \end{split}$$

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$$-E_{1}(\phi_{0}, F'G_{0}I_{1}\phi_{0}) + E_{1}^{2}(\phi_{0}, F'G_{0}F'\phi_{0}) -E_{1}^{2}(\phi_{0}, F''\phi_{0})\}.$$
 (7)

Thus it is shown that even if the energy appears quadratically in the fundamental equation, the perturbation energy can be calculated to any order by applying the similar procedure as in the nonrelativistic cases where the equation contains the energy linearly.

When there is no external field, all quantities become the functions of the relative energy-momentum, ε and $P = P_a = -P_b$. In particular, I_0 is the sole function of the relative momentum if it is instantaneous. Then we can define the three-dimensional wave function $\chi(P)$, following Salpeter, by

$$(\Lambda_{+}^{a}(\mathbf{P})\Lambda_{+}^{b}(\mathbf{P}) - \Lambda_{-}^{a}(\mathbf{P})\Lambda_{-}^{b}(\mathbf{P}))^{\chi}(\mathbf{P})$$
$$= \int d\varepsilon \, \psi_{0}(\mathbf{p}). \tag{8}$$

Here $\Lambda_{+}^{a}(\mathbf{P})$, $\Lambda_{-}^{a}(\mathbf{P})$, $\Lambda_{+}^{b}(\mathbf{P})$ and $\Lambda_{-}^{b}(\mathbf{P})$ are the usual Casimir operators; e.g., $\Lambda_{+}^{a}(\mathbf{P}) = [E_{a}(\mathbf{P}) + H_{a}(\mathbf{P})]/2E_{a}(\mathbf{P})$ where $E_{a}(\mathbf{P}) = +(m_{a}^{2} + \mathbf{P}_{a}^{2})^{1/2}$, and $m_{a} = m_{b} = m$ is supposed to have a small negative imaginary part.

The wave function $\chi(P)$ satisfies the equation

$$[E_0 - H_a(\mathbf{P}) - H_b(\mathbf{P})] \chi(\mathbf{P})$$

$$= -(2\pi) \int d^4k I_0(k) \psi_0(p+k). \tag{9}$$

From (8) and (9), the ϵ -dependence on $\psi_0(p)$ is uniquely determined as follows:

$$\psi_0(p) = -(2\pi i)^{-1}$$

$$\times \frac{E_0 - H_n(\mathbf{P}) - H_h(\mathbf{P})}{\left[\frac{1}{2}E_0 - H_c(\mathbf{P}) + \varepsilon\right]\left[\frac{1}{2}E_0 - H_b(\mathbf{P}) - \varepsilon\right]} \chi(\mathbf{P}),$$
(10)

where how to manage the poles in the integration over ϵ is decided according as $H_n(\mathbf{P})$ takes the value $+E(\mathbf{P})$ or $-E(\mathbf{P})$.

We define next $\psi^{\Delta}(p)$ by

$$\psi^{\Delta}(f) = -(2\pi i)^{-1}$$

$$\times \frac{E - H_a(\boldsymbol{P}) - H_b(\boldsymbol{P})}{\left[\frac{1}{2}E - H_a(\boldsymbol{P}) + \epsilon\right]\left[\frac{1}{2}E - H_b(\boldsymbol{P}) - \epsilon\right]} \chi(\boldsymbol{P}). \tag{11}$$

In (11) E is the eigenvalue of (1) and it is to be noted that $\int \psi_0(f) d\epsilon = \int \psi^{\Delta}(f) d\epsilon$. Then ψ^{Δ} is easily shown to satisfy

$$-(2\pi i)^{-1}(\Delta E)\chi = (F - I_0)\psi^{\Delta}(p). \tag{12}$$

Writing

$$\psi = \psi^{\Delta} + \psi_{\Delta} = \psi_0 + \psi_1^{\Delta} + \psi_2^{\Delta} + \cdots + \psi_{1\Delta} + \psi_{2\Delta} + \cdots + \psi_{1\Delta} + \psi_{2\Delta} + \cdots$$

we get from (1) and (12)

$$-(2\pi i)^{-1}(\Delta E)\chi = (\Delta I)\psi - (P_0 - I_0)\psi_{\Delta}$$
$$-[(\Delta E)F' + (\Delta E)^2F'']\psi_{\Delta}. \tag{13}$$

This is the fundamental equation of Salpeter's theory, from which the perturbation energies can be calculated. On the other hand, the zeroth order terms of (12) are nothing but the unperturbed equation (3). ψ_1^{Δ} and $\psi_{1\Delta}$ are obtained from (12) and (13) respectively, and each contains the auxiliary function $\chi(P)$, but their sum is free from $\chi(P)$ and coincides with $\psi_1(p)$ given by (6). Moreover, the equation (12) serves for showing the generality of our method of perturbation; as the first order terms of (12), we have

$$-(2\pi i)^{-1}E_1\chi = (F_0-I_0)\phi_1^{\Delta} + E_1F'\phi_0$$
,

which gives the relation

$$(\psi_0, F'\psi_0) = -(2\pi i)^{-1}(\psi_0, \chi). \tag{15}$$

 (ψ_0, χ) on the right hand side of (15) was normalized to unity by Salpeter. In this case, accordingly, E_1 becomes $-(2\pi i)\,(\psi_0,\,I_1\psi_0)$ in accordance with Salpeter's result. The wave function $\chi(\boldsymbol{P})$, therefore, seems not to play an essential role. From the second order terms of (12) and (15), we can also prove the relation

$$F_1(\psi_0, F''\psi_0) = -(\psi_0, F'\psi_1^{\Delta}),$$
 (16)

which reduces our second order perturbation energy (7) to Salpeter's.

It is thus obvious that our primitive perturbation theory, which is valid even in the presence of external field, includes completely Salpeter's method which can be used only in the absence of external field.

This perturbation theory can be applied, for example, to the calculation of the triplet splitting of He-atom, in which the nucleus is considered as an external field A_0 , the Coulomb interactions between two electrons as an unperturbed interaction I_0 , the exchange effects of a transverse photon between electrons as I_1 , and so on.

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- E. E. Salpeter, Phys. Rev. 87 (1952), 328. The direct application of Salpeter's method to the calculation of the hyperfine structure of the positronium has been made by R. Karplus and A. Klein, Phys. Rev. 87 (1952), 848.
- This assumption about an external field is permissible in most practical cases.
- Such terms may be frequently omitted in practical problems.