Progress of Theoretical Physics, Vol. 101, No. 1, January 1999

A Remark on the Smoothness of Critical KAM Curves in the Standard Mapping

Yoshihiro YAMAGUCHI^{*)} and Kiyotaka TANIKAWA^{*,**)}

Teikyo Heisei University, Ichihara 290-0193, Japan *National Astronomical Observatory, Mitaka 181-8588, Japan

(Received October 6, 1998)

The smoothness of KAM (Kolmogorov-Arnold-Moser) curves when approaching the critical point which constitutes the existence/non-existence boundary of a KAM curve is investigated in the standard mapping as a typical example of analytical twist mappings. We give numerical results indicating that the KAM curve at the critical point is C^1 , but its derivative is not a function of bounded variation. The fractal distribution function of mapped points on such KAM curves is caused by this unbounded variation.

§1. Introduction

We are interested in the smoothness of KAM curves in non-integrable hamiltonian systems. Even KAM curves in two-dimensional twist mappings exhibit complicated behavior. Birkhoff ¹) proved that homotopically non-trivial KAM curves of twist mappings of the annulus are Lipschitz continuous. This implies that these KAM curves are at least once differentiable almost everywhere in the sense of the Lebesgue measure.²) When we consider a one-parameter family of smooth twist mappings, we can generally expect that the smoothness of the KAM curves decreases as the maps deviate further and further from an integrable one.³) The breakup process of KAM tori was investigated extensively in non-twist systems.⁴) In the present paper, based on the above expectation, we consider a typical example of an analytical family of twist mappings, the standard mapping.

Let us introduce two notations. First, if a KAM curve with irrational rotation number ν exists for $a \leq a_c(\nu)$ and does not exist for $a > a_c(\nu)$, then $a_c(\nu)$ is called the critical parameter value for that KAM curve. If some KAM curve exists for $a \leq a_c$ and no KAM curve exists for $a > a_c$, then a_c is called the critical parameter value (for the family of maps). Second, the KAM curve at $a = a_c(\nu)$ or at $a = a_c$ is called the *critical curve*.

It is known that the critical curve is not analytic, but its length is finite. The motion restricted to a KAM curve is characterized by a one-dimensional circle mapping. Then, the following theorem by Denjoy^{5} establishes that the critical curve is at least C^1 .

Theorem 1-1 (Denjoy): If $G : S^1(\text{circle}) \to S^1$ is a C^1 diffeomorphism and its derivative is a function of bounded variation then G does not have a wandering interval.

^{*)} E-mail: yy-chaos@jb3.so-net.ne.jp

^{**)} E-mail: tanikawa@cc.nao.ac.jp

The critical curve is not $C^{2+\epsilon}$ ($\epsilon > 0$),⁶⁾ and the maximum smoothness of the critical curve with the golden mean rotation number is $C^{1+\gamma}$ ($\gamma < 1$).⁷⁾ The meaning of the notation $C^{1+\gamma}$ is explained in §5. The fractal structure of a distribution of mapped points on the critical curves appear, and the $f(\alpha)$ -spectrum⁸⁾ and information dimension for the critical curve are estimated.⁹⁾ This implies that the critical curve is not C^2 , since the distribution function is absolutely continuous if the critical curve is C^2 . Let us summarize several known results.

Property 1-2: The critical curve is $C^{1+\gamma}$ $(0 \le \gamma < 1)$.

However the authors feel there is some ambiguity in this statement. In fact, there exist examples of $C^{1+\gamma}$ diffeomorphisms that have wandering intervals.^{2),5)} Then, we cannot determine the existence or non-existence of a KAM curve by the Hölder continuity.

The numerical result obtained in this paper is that the critical curve is C^1 , but its derivative is not a function of bounded variation. This is an improved result compared with Property 1-2. Since the property of bounded variation of the derivative of a KAM curve does not hold at the critical value, the critical curve is at most C^1 , not $C^{1+\gamma}$ (see §5).

The organization of this paper is as follows. In order to study the transition to fractal curves from smooth curves, we measure the lengths of the graphs of KAM curves and of the related variables. To do so, we assume that the smoothness of KAM curves is C^r $(r \gg 1)$. Though the validity of this assumption is not assured, our results seem to justify it for $a < a_c(\nu)$. In §2, we review the standard mapping and the involutions. We introduce a circle mapping and derive the functional equations for the KAM curve and the circle mapping. In §3, the properties of KAM curves and the corresponding circle mapping are obtained, and all tools to study the structure of KAM curves are also obtained. Using such tools, we first estimate the critical value a_c and several critical values $a_c(\nu)$ in §4. In §5, the lengths mentioned above are calculated, and the change of smoothness is discussed. As a result, we derive our main result. The change of the distribution of mapped points on KAM curves is studied in §6. We also show that the fractal structure of the distribution function is caused by the unbounded variation of the derivative of the circle mapping. In §7, we make some remarks.

§2. Basic formulation of KAM curves

2.1. The standard mapping T(x, y) on a cylinder

The standard mapping T(x, y) $[-\infty < y < \infty, 0 \le x < 2\pi]$ on the surface of a cylinder is expressed as

$$y_{n+1} = y_n - a\sin x_n,\tag{1}$$

$$x_{n+1} = x_n + y_{n+1} (\text{Mod } 2\pi), \tag{2}$$

where a is a parameter and a > 0 is assumed. In the case that a < 0, we have the same expression using the coordinate change $x_n \mapsto x_n + \pi$. In the following, we let $f(x_n) = -a \sin x_n$.



Fig. 1. Several types of motion in phase space $(a = 0.8 \text{ (left)}, \frac{4}{3} \text{ (right)})$.

The standard mapping has area-preserving and orientation-preserving properties, and it is a typical twist mapping. The standard mapping has two fixed points (p and q). One of the fixed points, q = (0,0), is an elliptic point when 0 < a < 4, and is a saddle with reflection when a > 4. The other fixed point, $p = (\pi, 0)$, is always a saddle for a > 0. We can easily observe three types of motion in phase space (see Fig. 1): (1) periodic motion, (2) quasi-periodic motion, and (3) chaotic motion. Several important properties of the standard mapping are summarized in Ref. 10).

2.2. Involutionary forms of the standard mapping

The standard mapping T can be factorized into two involutions h and g:

$$T = h \circ g. \tag{3}$$

Two involutions h and g satisfy the properties

$$h \circ h = g \circ g = \mathrm{id.},\tag{4}$$

$$\det Dh = \det Dg = -1. \tag{5}$$

It is noted that the inverse map of T is given by $T^{-1} = g \circ h$. If the mapping functions have factorizations into involutions, the system possesses reversibility.¹¹ As the standard mapping possesses double reversibility,^{12),13} there are two types of involutionary forms.

[1] The first form:

$$h: y_{n+1} = y_n, \ x_{n+1} = -x_n + y_n, \tag{6}$$

$$g: y_{n+1} = y_n + f(x_n), \ x_{n+1} = -x_n.$$
(7)



Fig. 2. Symmetry lines in phase space. Two KAM curves $\Gamma_{1,2}$ exist above and below the x-axis, symmetrically. The explanations of the points u, v, \cdots are given in the caption of Table I.

The set of fixed points of h and g give the symmetry lines illustrated in Fig. 2:

$$h: y = 2x, \ y = 2(x - \pi),$$
 (8)

$$g: x = 0, \ \pi. \tag{9}$$

These symmetry lines of the first form show the left and right symmetry of KAM curves. We use this property in the following discussion. [2] The second form:

$$h': y_{n+1} = -y_n, \ x_{n+1} = x_n - y_n, \tag{10}$$

$$g': y_{n+1} = -y_n - f(x_n), \ x_{n+1} = x_n.$$
(11)

The symmetry lines are also obtained (see Fig. 2) for this case:

$$h': y = 0, \tag{12}$$

$$g': y = \frac{a}{2}\sin x. \tag{13}$$

These symmetry lines show the symmetrical structure of two KAM curves existing above and below the x-axis. If one KAM curve (Γ_1) with rotation number ν exists above the x-axis, another KAM curve (Γ_2) with rotation number $-\nu$ must exist at the symmetrical position corresponding to Γ_1 below the x-axis. Such a situation is illustrated in Fig. 2. The mapping T is invariant under the transformation $y \to y \pm 2\pi$. Then, the KAM curve with rotation number $1-\nu$ constructed by the transformation $y \to y - 2\pi$ for Γ_2 exists. Two KAM curves with respective rotation numbers ν and $1-\nu$ have the same properties. As a result, it is sufficient to study the properties of KAM curves with $0 < \nu < 0.5$.

2.3. Functional equations for KAM curves

We shall derive various properties of KAM curves and the corresponding circle mapping assuming they are C^r $(r \gg 1)$.

A KAM curve is a graph of a Lipschitz function $y_n = F(x_n)$. Here we introduce the circle map, $x_{n+1} = G(x_n)$, which defines the relation between x_n and x_{n+1} on the KAM curve. Using the mapping equations, we can derive the functional equations for F(x) and G(x) and give the relation between F(x) and G(x):

$$F(x + F(x) + f(x)) = F(x) + f(x),$$
(14)

$$G(x) + G^{-1}(x) = 2x + f(x),$$
(15)

$$G(x) = x + F(x) + f(x),$$
 (16)

where the suffix n is omitted.

Hereafter we use the abbreviations

$$\xi_n = \frac{dF(x_n)}{dx_n}, \qquad \zeta_n = \frac{dG(x_n)}{dx_n}, \qquad \epsilon_n = \frac{df(x_n)}{dx_n}.$$

Here ξ_n is the slope of the KAM curve at x_n and ζ_n is the gradient of $G(x_n)$ at x_n .

Differentiating Eqs. (14) and (15) with respect to x_n , we also have the functional equations for ξ_n and ζ_n ,

$$\xi_{n+1} = \frac{\xi_n + \epsilon_n}{1 + \xi_n + \epsilon_n},\tag{17}$$

$$\zeta_{n+1} = -\frac{1}{\zeta_n} + 2 + \epsilon_{n+1}.$$
 (18)

§3. Some properties of KAM curves

3.1. Symmetrical structure

First, we give a property of motion of mapped points restricted on KAM curves encircling the cylinder.

Proposition 3-1: Any KAM curve encircling the cylinder is invariant under g and hence invariant under h.

Proof: If we prove that any KAM curve is invariant under g, its invariance under h follows from $T = h \circ g$. Since the KAM curve intersects the symmetry lines of g, let us take a point u of intersection points. Let $O(u) = \{u_i | u_i = T^i u, i \in \mathbb{Z}\}$ be its orbit. By reversibility, $g(u_n) = g \circ T^n u = T^{-n} \circ g(u) = T^{-n} u = u_{-n}$. This means that the orbit of u is invariant under g.

The orbits O(u) are dense in the KAM curve. Let w be an arbitrary point of this curve. Then there exists a sub-sequence u_{n_i} $(n_i \to \infty \text{ as } i \to \infty)$ which converges at w. By continuity of the reversibility, g(w) is on the KAM curve, and consequently its orbit is invariant under g.

Let us take two positions $l = (x_l, y_l)$ and $r = (x_r, y_r)$ satisfying the relation r = h(l) or r = g(l) on the specified KAM curve. Using Eqs. (6) and (7), we can derive the relations of the slopes at l and r. Here ξ_l and ξ_r represent these slopes. In the following expressions, we can exchange the two suffices l and r.

[1] The relation with respect to the symmetry lines $(y = 2x \text{ and } y = 2(x - \pi))$ of h is given by

$$\xi_r = \frac{\xi_l}{\xi_l - 1}.\tag{19}$$

[2] The relation with respect to the symmetry lines $(x = 0 \text{ and } x = \pi)$ of g is given by

$$\xi_r = -\xi_l - f'(x_l).$$
⁽²⁰⁾

Equations (19) and (20) give information on the slopes of KAM curves at their intersection points with the symmetry lines. Using the relation $\xi_l = \xi_r$ at the intersection points, we have the following results:

(1) At the intersection points of all KAM curves and the symmetry lines of h,

$$\xi = 0. \tag{21}$$

(2) At the intersection points of all KAM curves and the symmetry lines of g,

$$\xi(0) = -\frac{1}{2}f'(0) = \frac{a}{2},\tag{22}$$

$$\xi(\pi) = -\frac{1}{2}f'(\pi) = -\frac{a}{2}.$$
(23)

Take a point A = (x, y) on the KAM curve. The point B = (x', y') = g(A) also is located on the same KAM curve, due to Proposition 3-1. Using the relations x' = -x, y' = y + f(x), y = F(x) and y' = F(x'), Eq. (14) can be rewritten in the form

$$F(x) - F(-x) = -f(x).$$
 (24)

Let us divide the function F(x) into even and odd parts:

$$F(x) = F_{\text{even}}(x) + F_{\text{odd}}(x).$$
(25)

Then we have

$$F_{\rm odd}(x) = -\frac{1}{2}f(x).$$
 (26)

We need to determine the even part $F_{\text{even}}(x)$ to obtain the full expression of the KAM curve. This problem will be discussed later.

Here we have an important result on the slope of the KAM curves. **Proposition 3-2**: The relation $\xi(x) < 1$ holds for all KAM curves.

Proof: Suppose that there exists a point A with $\xi > 1$ on the KAM curve. But by Eq. (19), the slope at B = h(A) is positive. Let us take a small arc AA' on the KAM curve in the vicinity of A where the x-coordinate of A' is larger than that of A. Note that the y-coordinate of A' is larger than that of A since the slope at A is positive. The involution h maps this arc to an arc BB' where both x and y-coordinates of B'

6

Table I. The slope of KAM curves at several points. u is the intersection point of the KAM curve and y-axis, v is the intersection point of the KAM curve and $x = \pi$, s is the intersection point of the KAM curve and the line y = 2x, and t is the intersection point of the KAM curve and the line $y = 2(x - \pi)$.

Name	Position in phase space	Slope
u	$\stackrel{\mathrm{def}}{=} (0, \alpha)$	$\frac{a}{2}$
Tu	(lpha, lpha)	$-rac{a}{2-a}$
$T^{-1}u$	$(-\alpha(+2\pi), \alpha - a\sin\alpha)$	$a\cos\alpha + \frac{a}{2-a}$
v	$\stackrel{\mathrm{def}}{=}(\pi,\beta)$	$-\frac{a}{2}$
Tv	$(\pi+eta,eta)$	$\frac{a}{2+a}$
$T^{-1}v$	$(\pi - eta, eta + a \sin eta)$	$-a\cos\beta - \frac{a}{2+a}$
s	$\stackrel{\rm def}{=} (\gamma, 2\gamma)$	0
Ts	$(3\gamma - a\sin\gamma, 2\gamma - a\sin\gamma)$	$\frac{a\cos\gamma}{a\cos\gamma-1}$
$T^{-1}s$	$(-\gamma(+2\pi), 2\gamma - a\sin\gamma)$	$a\cos\gamma$
t	$\stackrel{\text{def}}{=} (\delta + \pi, 2\delta)$	0
Tt	$(3\delta + a\sin\delta + \pi, 2\delta + a\sin\delta)$	$\frac{a\cos\delta}{a\cos\delta+1}$
$T^{-1}t$	$(\pi - \delta, 2\delta + a\sin\delta)$	$-a\cos\delta$

are larger than those of B, since the slope of the KAM curve at B is positive. The three points, A', the intersection point (C) of the KAM curve and the symmetry line of h, and B' are located in this order on the KAM curve. Then the arc CB' of the KAM curve must contain the point (D) where the slope diverges. If the slope at A is equal to 1, the value of the slope at B diverges. Then the KAM curve does not satisfy the Lipschitz condition. This contradicts the result obtained by Birkhoff.¹⁾

In Table I, we list several properties of KAM curves (see Fig. 2) in phase space. These properties are determined from the results mentioned above and obtained in §3.6.

3.2. Property of the circle mapping G(x)

Here we show the symmetrical structure of G(x).

Proposition 3-3: The graph of G(x) is symmetric with respect to the lines $y = -x(+2\pi)$. Then the relation $G(x) = -G^{-1}(-x) \pmod{2\pi}$ holds (see Fig. 3).

Proof: We give proof in terms of the relation between the phase space and the circle mapping. Allow us to list the correspondence of the phase space and the circle mapping:



Fig. 3. The circle mapping G(x) and its symmetry structure.

Operation in Phase Space	\Leftrightarrow	Operation on the Circle
g	\Leftrightarrow	$x\mapsto -x$
h	\Leftrightarrow	G(-x)
T	\Leftrightarrow	G(x)

Then $-G^{-1}(-x)$ is considered as the composition of three successive operations, (1) $x \mapsto -x$, (2) G^{-1} , (3) $x \mapsto -x$. These operations are transformed into the following operations in phase space:

(1) Operate g, (2) Operate T^{-1} , (3) Operate g. As a result, we have

$$g \circ T^{-1} \circ g = g \circ g \circ h \circ g = h \circ g = T.$$

This final operation T is transformed into the operation G(x) of the circle mapping. Then Proposition 3-3 follows.

It is worth noting that Proposition 3-3 is derived by direct calculation using Eq. (26). This fact implies that the reversibility $(T \circ g = g \circ T^{-1})$ determines the expression of $F_{\text{odd}}(x)$ (see Eq. (26)), and gives the symmetry of G(x) stated in Proposition 3-3.

Using Proposition 3-3, we now give several properties of the circle mapping.

3.3. Properties of G_{odd} and the functional equation for G_{even}

We give explicit expressions of G_{odd} and G_{odd}^{-1} and derive the functional equation for G_{even} .

Property 3-4: $G_{\text{odd}}(x) = G_{\text{odd}}^{-1}(x) = x + \frac{1}{2}f(x) \pmod{2\pi}.$

Proof: Combining the relation of Proposition 3-3 with Eq. (15), we have

$$G(x) - G(-x) = 2x + f(x).$$
 (27)

Let us divide G(x) into even and odd parts:

$$G(x) = G_{\text{even}}(x) + G_{\text{odd}}(x).$$
(28)

As a result, Eq. (27) gives the expression of the odd function G_{odd} :

$$G_{\text{odd}}(x) = x + \frac{1}{2}f(x).$$
 (29)

Using the same method, we have

$$G_{\text{odd}}^{-1}(x) = x + \frac{1}{2}f(x).$$
 (30)

Thus we have Property 3-4.

Using Eq. (15) and Property 3-4, the relation between $G_{\text{even}}(x)$ and $G_{\text{even}}^{-1}(x)$ is derived:

$$G_{\text{even}}(x) + G_{\text{even}}^{-1}(x) = 0.$$
 (31)

Combining the relation $G^{-1}(G(x)) = x$, Eq. (31) and Property 3-4, the functional equation for $G_{\text{even}}(x)$ is obtained:

$$G_{\text{even}}(G(x)) = G_{\text{even}}(x) + \frac{1}{2}f(x) + \frac{1}{2}f(G(x)).$$
(32)

The initial condition (for example, $G_{\text{even}}(0)$) to solve Eq. (32) is obtained as follows. Since $G(0) = G_{\text{even}}(0) + G_{\text{odd}}(0) = G_{\text{even}}(0)$ and $F(0) = G(0) = G_{\text{even}}(0)$, the value of $G_{\text{even}}(0)$ is determined by the initial condition (0, F(0)) in phase space.

Applying the procedure mentioned above to the circle mapping $G(x + \pi)$, we have the same results for $G_{\text{even}}(x)$. Thus we have Property 3-5. **Property 3-5:** The graph of the even part, the function $G_{\text{even}}(x)$, has two symmetry

lines, x = 0 and $x = \pi$.

We can derive the functional equation for $G'_{\text{even}}(x)$,

$$G'_{\text{even}}(G(x)) = \frac{1}{2}f'(G(x)) + \frac{G'_{\text{even}}(x) + \frac{1}{2}f'(x)}{G'_{\text{even}}(x) + \frac{1}{2}f'(x) + 1}.$$
(33)

We can also derive the functional equation for the second derivative $G''_{\text{even}}(x)$,

$$G_{\text{even}}''(G(x)) = \frac{1}{2}f''(G(x)) + \frac{G_{\text{even}}''(x) + \frac{1}{2}f''(x)}{\left(G_{\text{even}}'(x) + \frac{1}{2}f'(x) + 1\right)^3}.$$
 (34)

Differentiating Eq. (34), we can derive the functional equations for higher order derivatives $G_{\text{even}}^{(n)}(x)$ used in §5. Here we omit these expressions. We must comment on the initial values to solve $G_{\text{even}}^{(n)}(x)$. Symmetry gives the following initial conditions:

$$G_{\text{even}}^{(2n-1)}(0) = G_{\text{even}}^{(2n-1)}(\pi) = 0. \qquad (n \ge 1)$$
(35)

Strictly speaking, we have no information on the initial conditions to solve $G_{\text{even}}^{(2n)}(x)$ $(n \ge 1)$. For this reason, we use the spline interpolation to determine $G_{\text{even}}^{(2n)}(0)$ $(n \ge 1)$. A detailed explanation of this procedure is given in Appendix D.

3.4. Structure change of KAM curves

To study the structure of a KAM curve characterized by an irrational rotation number ν , we must plot the figures of G(x) (or $G_{\text{even}}(x)$) and its derivative G'(x)(or $G'_{\text{even}}(x)$). To do so, we solve the functional equations for G_{even} and G'_{even} numerically. This procedure is described in detail in Appendix A.

Note that the most important calculation in this procedure is to determine the intersection point of the KAM curve with the symmetry line. Typical examples obtained through numerical results are shown in Fig. 4. The symmetry of G and the several relations mentioned in §§3.2 and 3.3 are confirmed. As for the structure of $G'_{\text{even}}(x)$, we observe the transition from a smooth curve to a notched curve when the parameter a is increased toward the critical value $a_c(\nu)$. This type of transition is also found in KAM curves with other irrational rotation numbers, for example, the silver mean $\sqrt{2} - 1$ (see Fig. 4(b)). Detailed discussion of the singular structure of $G'_{\text{even}}(x)$ is given in §5.

3.5. Estimation of maximum and minimum values of G'(x)

Here we determine the maximum value of G'(x). To do so, the property of mapping function Eq. (18) is used. This is different from the method used by Herman,²⁾ but our estimation agrees with his result. If the maximum value of $\zeta(x) (= G'(x))$ is M, then the minimum value ζ_{\min} is $\zeta_{\min} = \frac{1}{M}$, due to the symmetry of G(x) (Proposition 3-3). The position with the maximum value M exists at position symmetric to that with the minimum value $\frac{1}{M}$, where the symmetry lines are $y = -x(+2\pi)$. We now determine an estimate of M. Equation (18) with a = 1 is illustrated in Fig. 5. The lower (upper) curve represents the mapping function with $\epsilon_n = 1$ (-1). All mapping functions exist in the region bounded by these two curves. Then, if there exists an integer n satisfying $\zeta_n > \zeta_2$, the sequence $\{\zeta_{-i}\}$ ($i \ge n+1$) diverges. As a result, we have the maximum value $M = \zeta_2$:

$$M = \frac{2+a+\sqrt{a^2+4a}}{2}.$$
 (36)

The following inequality must hold for any integer n in order for the KAM curve to exist:

$$\frac{1}{M} \le \zeta_n \le M. \tag{37}$$



Fig. 4. Figures (a) and (b) represent the results for the KAM curves with rotation numbers of the golden mean (a) and the silver mean (b). The upper figures display $G_{\text{even}}, G_{\text{odd}}$, and G, and the lower figures display $G'_{\text{even}}, G'_{\text{odd}}$ and G'. a=0.92 (left), and 0.97 (right) in (a), and a=0.92 (left), and 0.95 (right) in (b).



Fig. 5. Mapping function of $\zeta_n \mapsto \zeta_{n-1}$ (a = 1).

Herman estimated the minimum value ζ_{\min} as

$$\frac{2+a-\sqrt{a^2+4a}}{2} \le \zeta_{\min} \le 1-\frac{a}{2}.$$
 (38)

According to our numerical results given in Fig. 4, we have

Minimum value
$$\approx 1 - \frac{a}{2}$$
,
Maximum value $\approx \frac{1}{1 - \frac{a}{2}}$.

To obtain a better estimate of the maximum and minimum values compared with Eq. (37) is an open problem.

Finally we comment on the meaning of Eq. (37). If Eq. (37) does not hold, the condition of Lipschitz continuity is broken and then the KAM curve does not exist. Using Eq. (37), we can estimate the critical value a_c (see §4).

3.6. Summary of the properties of G(x)

Finally we summarize other properties of G(x). [1] Let A denote the position of G(0) in Fig. 3. Then,

$$\zeta(A) = 1 - \frac{a}{2}, \qquad (39)$$

$$\zeta(B) = \frac{1}{1 - \frac{a}{2}},\tag{40}$$

where A = G(B). The first relation gives Eq. (22) in phase space.

[2] At the intersection points C_1 and C_2 of G(x) and $y = -x(+2\pi)$, the gradient of G(x) is 1 due to Proposition 3-3.

[3] Using the same method as in [1], we have the slope at D in Fig. 3,

$$\zeta(\pi) = 1 + \frac{a}{2} \,. \tag{41}$$

This relation gives Eq. (23) in phase space.

§4. Estimation of critical values a_c and $a_c(\nu)$

Using the criterion (37), we determine the critical value a_c . We consider the sequence of mapped points $\{T^{-n}u\}$ $(n \ge 1)$ in phase space. Hereafter we iterate Eq. (18) to calculate the critical value. The mapping function (18) gives ζ_{-1} (slope of G) at $G^{-1}(0)$ $(T^{-1}u$ in phase space):

$$\zeta_{-1} = \frac{1}{2 - a - \left(1 - \frac{a}{2}\right)} \ . \tag{42}$$

 ζ_{-1} must satisfy the relation

$$\frac{1}{M} \le \frac{1}{1 - \frac{a}{2}} \le M. \tag{43}$$

The second inequality gives the condition for the existence of KAM curves, $a \leq \frac{4}{3}$. The first inequality does not give a condition. From this, we obtain Proposition 4-1: **Proposition 4-1**: There are no KAM curves at $a > \frac{4}{3}$.¹⁴⁾

Using information regarding ζ_{-n} , we obtained the improved critical values given in Table II. The detailed procedure is explained in Appendix B. This procedure gives a sufficient condition for a_c . In fact, we decrease the value of a from the parameter region where there are no KAM curves. As a result, we find that there are no KAM curves at $a \ge 0.972$. Using the residue method, Greene ¹⁵

Table II. The critical value a_c .

n	a_n	n	a_n
3	1.198	89	0.97908
5	1.116	144	0.97619
8	1.0579	233	0.97429
13	1.0250	377	0.97335
21	1.0039	610	0.97269
34	0.9914	987	0.97219
55	0.9837	1597	0.97196

obtained that the KAM curve exists at $a < 0.971635\cdots$. Using the interval arithmetic, Mackay-Percival [M-P] ¹⁶ determined that there are no KAM curves at $a > \frac{63}{64}$. According to the method of M-P, we need a huge number of iterations. As we use the functional equation (18), we have a good estimation in spite of the number of iterations up to 1597-times. Our criterion for the non-existence of a KAM curve is slightly different from the Lipschitz criterion used by M-P.

Using the property that if an orbit goes slowly and quickly then there are no KAM curves between the slower and faster rates, the critical value was estimated as a = 0.9718 by Jungreis.¹⁷ But it is not easy to apply this method to calculate $a_c(\nu)$.

Rotation Number	$a_c(u)$	α (±0.002)	β (±0.002)
$(\sqrt{5}-1)/2$	0.9717	0.292	0.072
$\sqrt{2}-1$	0.9575	0.302	0.124
$(\sqrt{13}-3)/2$	0.8909	0.327	0.116
$\sqrt{7}-2$	0.8809	0.350	0.167
$\sqrt{3}-1$	0.8762	0.327	0.165
$\sqrt{6}-2$	0.8735	0.357	0.185
$\sqrt{5}-2$	0.8048	0.350	0.185
$\sqrt{8}-2$	0.6842	0.423	0.266
$\sqrt{10} - 3$	0.6475	0.434	0.254

Table III. The critical values for several KAM curves and two exponents α and β . Note that the KAM curve with rotation number $\sqrt{2} - 1$ (for example) do not exist at a = 0.9575.

Using the mapping function (18) and the criterion (37), we can estimate the critical value $a_c(\nu)$ for the KAM curve (chief KAM curve) with any irrational rotation number ν . The procedure is described in Appendix C. In this procedure, we need a precise position of the KAM curve. As we do not have the exact position numerically, we assume that KAM curves in the ϵ -neighborhood of the chief KAM curve have the same stability as the chief KAM curve. This implies that nearby KAM curves exist (disappear) if the chief KAM exists (disappears). If we take the limit $\epsilon \to 0$, we obtain $a_c(\nu)$, but its value is an upperbound (see also the caption in Table III). In Table III, several critical values are given. From Table III, we also see that if the KAM curve with rotation number equal to the golden mean is the last KAM curve in the standard mapping, there are no KAM curves at $a \geq 0.9717$.

§5. Smoothness of KAM curves

In order to study the singular behavior of KAM curves in the vicinity of their critical values, we directly calculate the length of the graph of $G_{\text{even}}(x)$ and that of the graph of the *n*-th derivative $G_{\text{even}}^{(n)}(x)$. For example, the length $L(G_{\text{even}}(x))$ of $G_{\text{even}}(x)$ is calculated in terms of the following two formulas:

$$\mathbf{FN}: \quad L(G_{\text{even}}(x)) = \sum_{i=1}^{n+1} \Delta x_i \sqrt{1 + \left(\frac{\Delta G_{\text{even}}(x_i)}{\Delta x_i}\right)^2}, \tag{44}$$

FD:
$$L(G_{\text{even}}(x)) = \sum_{i=1}^{n+1} \Delta x_i \sqrt{1 + (G'_{\text{even}}(x_i))^2}.$$
 (45)

Here we explain the notation in **FN**. We calculate the sequences $\{x_i\}$ and $\{G_{\text{even}}(x_i)\}$ $(i = 1, \dots, n)$, and sort the sequence $\{x_i\}$ with $\{G_{\text{even}}(x_i)\}$ in order of increasing values. The two sorted sequences are renamed $\{x_i\}$ and $\{G_{\text{even}}(x_i)\}$. Note that $\Delta x_i = x_i - x_{i-1}, x_0 = 0, x_{n+1} = 2\pi, \Delta G_{\text{even}}(x_i) = G_{\text{even}}(x_i) - G_{\text{even}}(x_{i-1})$, and $G_{\text{even}}(0) = G_{\text{even}}(2\pi)$ (the initial value). The same notation is also used in **FD**.

If the length estimated by **FN** is equal to that estimated by **FD**, the function $G_{\text{even}}(x)$ is considered once differentiable. Using the same method, we calculate

Table IV. The lengths estimated by two formulas. The asterisk means that we do not calculate the length, and a blank entry indicates that the calculated lengths do not converge. The explanation is given in Appendix D.

Formula	a	$L(G_{\text{even}})$	$L(G'_{\mathrm{even}})$	$L(G''_{even})$	$L(G_{\text{even}}^{(3)})$	$L(G_{\text{even}}^{(4)})$	$L(G_{\mathrm{even}}^{(5)})$
FN	0.9500	6.424	7.496	61.09	3.744E + 03	4.464E + 05	8.119E + 08
FD	0.9500	6.424	7.496	61.09	3.744E + 03	4.464E + 05	*
FN	0.9600	6.431	7.755	114.9	1.364E + 04	3.119E + 06	1.072E + 09
FD	0.9600	6.431	7.755	114.9	1.364E + 04	3.119E + 06	*
FN	0.9670	6.436	8.130	295.5	9.162E + 04	5.464E + 06	1.072E + 09
FD	0.9670	6.436	8.130	295.5	9.162E + 04	5.464E + 06	*
FN	0.9700	6.438	8.540	862.4	7.881E + 05		
FD	0.9700	6.438	8.540	862.4			*
FN	0.9715	6.438	10.95				
FD	0.9715	6.438					*

 $L(G_{\text{even}}^{(n)}(x))$ $(n = 1, \dots, 5)$ and check the differentiability.

When we use the formula **FN**, we need the data of $G_{\text{even}}(x_n)$ calculated by Eq. (32). On the other hand, for the formula **FD**, we use the data of $G'_{\text{even}}(x_n)$ calculated by Eq. (33). As the initial values of $G^{(2n)}_{\text{even}}(x_n)$ are not determined theoretically, we calculate them using a numerical method. This procedure is described in Appendix D. Once the data of $G^{(n)}_{\text{even}}(x_n)$ are obtained, it is easy to calculate its length.

The numerical results for the golden mean KAM curve are given in Table IV. The two lengths are in agreement even in the vicinity of critical value. The same result is obtained for the KAM curves with rotation numbers $\sqrt{2} - 1$ (silver mean) and $(\sqrt{13} - 3)/2$. As a result, we have the following conjecture:

Conjecture 5-1: The function $G_{\text{even}}(x)$ is C^r $(r \gg 1)$ at $a < a_c(\nu)$. Smoothness is suddenly lost at $a_c(\nu)$.

Next we show the parameter dependence of lengths of $G_{\text{even}}^{(n)}(x)$ $(n \ge 0)$ for KAM curves with the rotation numbers of the golden mean and of the silver mean (see Fig. 6). We list three numerical results:

(1) The length of the graph of $G_{\text{even}}(x)$ slightly increases when the parameter a increases to the critical value, but the length at the critical value is finite.

(2) The length of the graph of $G'_{\text{even}}(x)$ diverges very slowly as $(a_c(\nu) - a)^{-\alpha}$ with $\alpha = 0.056$ for the golden mean and $\alpha = 0.058$ for the silver mean.

(3) The length of the graph of $G_{\text{even}}^{(n)}(x)$ $(n \ge 2)$ diverges very rapidly as $(a_c(\nu) - a)^{-\alpha}$ with $\alpha = n - 1 + \epsilon$ $(0 < \epsilon < 1)$.

According to these results and to the composition of G'(x) constructed by $G'_{\text{odd}}(x)$ (the bounded variation function) and $G'_{\text{even}}(x)$, we arrive at the property that the function G(x) is once differentiable, but the graph of G'(x) does not have length at the critical value. Then G'(x) is not a function of bounded variation at the critical value.

Next we show that the critical curve is not $C^{1+\gamma}$ $(0 < \gamma < 1)$. Here we define the γ -Hölder derivative of $\xi(x_n)$:



$$\eta_{\gamma}(x) = \begin{cases} \lim_{x' \to x^{+}} \frac{\xi(x') - \xi(x)}{(x' - x)^{\gamma}}, \\ \lim_{x' \to x^{-}} \frac{\xi(x') - \xi(x)}{-|x' - x|^{\gamma}}. \end{cases}$$
(46)

If the γ -Hölder derivative of $\xi(x)$ is equal to zero at all points on the KAM curve,



Fig. 6. Figure (a) ((b)) displays the lengths $L(G_{\text{even}}^{(n)}(x))$ ($0 \le n \le 5$) of the graph of $G_{\text{even}}^{(n)}(x)$ for the golden(silver) mean KAM curve. α_n is the exponent in the expression $L(G_{\text{even}}^{(n)}(x)) \propto (a_c - a)^{-\alpha_n}$. (a) $\alpha_1 = 0.056$, $\alpha_2 = 1.01$, $\alpha_3 = 2.08$, $\alpha_4 = 3.13$ and $\alpha_5 = 4.19$. (b) $\alpha_1 = 0.058$, $\alpha_2 = 1.04$, $\alpha_3 = 2.15$, $\alpha_4 = 3.21$ and $\alpha_5 = 4.34$.

the KAM curve is C^r $(r \ge 2)$. If the KAM curve is $C^{1+\gamma}$, then the γ -Hölder derivative of $\xi(x)$ has a bounded value away from zero at some position x on the KAM curve. Using Eq. (17), we have the following relation between $\eta_{\gamma}(x)$ and $\eta_{\gamma}(G(x))$:

Y. Yamaguchi and K. Tanikawa

$$\eta_{\gamma}(G(x)) = \frac{\eta_{\gamma}(x)}{(G'(x))^{2+\gamma}}.$$
(47)

Then we have

$$\log \frac{\eta_{\gamma}(x_n)}{\eta_{\gamma}(x_0)} = -(2+\gamma) \sum_{i=0}^{n-1} \log G'(x_i) , \qquad (48)$$

where $x_0 = x$ and $x_i = G^i(x)$. Since G'(x) at $a = a_c(\nu)$ is not a function of bounded variation, the sum on the r.h.s. of Eq. (48) is not bounded, and $\eta_{\gamma}(x_n)$ is not bounded. This means that the critical curve is not $C^{1+\gamma}$ ($0 < \gamma < 1$). Then the critical curve is at most C^1 . Combining the above results, we have the main result that the critical curve is C^1 , but its derivative is not a function of bounded variation. This result implies that the graph of F'(x) of the critical curve has infinitely many folds.

§6. Distribution of mapped points on critical curves

Here we study the structure change of the density distribution $\rho(x)$ $(0 \le x < 2\pi)$ of mapped points on a KAM curve. As $\rho(x)$ is invariant under T, the Frobenius-Perron functional equation holds for $\rho(x)$:

$$\rho(x_{n+1}) = \frac{\rho(x_n)}{G'(x_n)}.$$
(49)

Using Eq. (49) with $\rho(0) = 1$ (for example), T, and Eq. (18) for G'(x), we can easily determine the structure of $\rho(x)$.

Equation (49) gives the relation

$$\log \frac{\rho(x_n)}{\rho(x_0)} = -\sum_{i=0}^{n-1} \log G'(x_i) \,. \tag{50}$$

Since G'(x) is not a function of bounded variation at the critical value, the ratio $\rho(x_n)/\rho(x_0)$ in Eq. (50) is not defined. This shows that the function $\rho(x)$ has a fractal property only at $a = a_c(\nu)$.

Next we give the symmetry structure. **Property 6-1:** $\rho(x) = \rho(2\pi - x)$. **Proof:** Using Eq. (49), we have

$$\frac{\rho(2\pi - x)}{G'(2\pi - x)} = \rho(G(2\pi - x)) = \rho(G^{-1}(x)).$$
(51)

Here we used the relation of Proposition 3-3 to derive the last equality. Combining Eq. (51) and the relation $G'(2\pi - x) = (G^{-1}(x))'$, we have the relation

$$\rho(2\pi - x) = \rho(G^{-1}(x))G'(2\pi - x) = \rho(G^{-1}(x))(G^{-1}(x))' = \frac{\rho(G^{-1}(x))}{G'(G^{-1}(x))} = \rho(x).$$
(52)



Fig. 7. The density distribution functions $\rho(x)$ for several rotation numbers. The phase plot around the KAM curve is shown in the upper part of each figure. The horizontal axis is the x-axis ($0 \le x < 2\pi$). (a) golden mean (a = 0.97), (b) silver mean (a = 0.888), (c) ($\sqrt{13} - 3$)/3 (a = 0.888), and (d) $\sqrt{7} - 2$ (a = 0.875).

This completes the proof.

Property 6-2: $\rho'(0) = \rho'(\pi) = 0$ at $a < a_c(\nu)$. **Proof:** This is trivial since there exist G'(x) and G''(x) at $a < a_c(\nu)$.

In Fig. 7, the distributions for four rotation numbers in the vicinity of the critical values are shown, and the KAM curve itself and several types of motion near the KAM curve are also illustrated. With our numerical results, we can list the universal properties which hold for all KAM curves in the vicinity of the critical values (see Fig. 8):

(1) Increasing a toward $a_c(\nu)$, the distribution at u decreases toward zero as $\rho(u) \propto (a_c(\nu) - a)^{\alpha}$ ($\alpha > 0$). Then, a steep-walled valley appears at points $T^n u$ ($n = 0, \pm 1, \cdots$).



Fig. 8. The increasing and decreasing of distribution functions $\rho(x)$ at u, v, s and t.

(2) Increasing a toward $a_c(\nu)$, the distributions at ν, s , and t diverge as $\rho \propto (a_c(\nu) - a)^{-\beta}$ ($\beta > 0$). The peak appears at points $T^n \nu, T^n s$, and $T^n t$ ($n = 0, \pm 1, \cdots$).

Two exponents α and β are obtained in Table III. As a result, we have a conjecture:

Conjecture 6-3: The KAM curve with golden mean rotation number corresponds to the minimum values of the two exponents α and β among all KAM curves.

This conjecture corresponds to the fact that the KAM curve with golden mean rotation number is the most stable and persistent one. Hunt et al.⁹⁾ calculated the information dimension of the critical KAM curves and reported that the KAM curve with the golden mean rotation number has the maximum information dimension less than 1. But the relation between our result and the last KAM curve is not understood. We believe that there exists a relation between the information dimension and two exponents α and β .

Our main result in §5 together with the results in this section give a new interpretation of the breakup of a KAM curve. The property of bounded variation of G'(x) guarantees the existence of KAM curves. At $a < a_c(\nu)$, the bounded variation of G'(x) holds and KAM curves are very smooth $(C^r \ (r \gg 1))$, contrary to our expectation. Since the property of bounded variation for $G'_{\text{even}}(x)$ does not hold at the critical value, holes, i.e., the points of zero density, at $\{T^{\pm n}u \ (n = 0, 1, \cdots)\}$ may appear at $a = a_c(\nu)$. Holes are mapped to holes. As a result, on the critical KAM curve, there appear two kinds of motion. One is the motion on the full Lebesgue measure set, except the holes. This is not the so-called Aubry-Mather set,¹⁸ but may be its predecessor to $a > a_c(\nu)$. The other is a set of holes. This is a countable invariant set. This may be a predecessor to the wandering points in the circle map, and is a candidate of gaps of the Aubry-Mather set for $a > a_c(\nu)$.

§7. Concluding remark

We now comment on the origin of the self-similar-like structure of $\rho(x)$. From Fig. 7, we observe that the distribution increases as a whole if saddles exist in the neighborhood of the KAM curve, and it decreases if large elliptic islands exist. For example, the elliptic islands with rational rotation numbers characterized by the Farey series accumulate toward u. This accumulation cannot be observed around v, s, and t. From Eq. (49), we find that the change of local structure of $\rho(x)$ is influenced by the absolute value of G''(x) corresponding to the curvature of the KAM curve. The curvature is strongly affected by the existence of periodic points around the KAM curve. In order to make clear this property, we have to elucidate the accumulation structure of periodic points to the KAM curve and their stability without using the assumption of scaling. The reason is the following. Two interpretations of the hierarchical structure around KAM curves have been proposed, the renormalization group theory by Greene, Mackay, Percival, and Shenker-Kadanoff,¹⁹⁾ and the theory proposed by Morbidelli and Giorgilli (MG).²⁰⁾ The MG theory gives the existence of a neighborhood almost completely full of slave KAM tori around a chief KAM torus. In Ref. 6), the same conclusion was obtained. Recent numerical calculations by Lega-Froeschlé²¹⁾ support the MG theory.

Acknowledgements

We are indebted to Professor N. Aoki for many discussions and suggestions.

Appendix A

We describe the procedure to solve the functional equations for G_{even} and G'_{even} . We can use Procedure I when the initial condition is taken on the $x = \pi$ -axis. **Procedure I:** Calculation of $G_{\text{even}}(x)$ and $G'_{\text{even}}(x)$.

[Step 1] This step is used to find the intersection point $(0, y_0)$ of a KAM curve with the y-axis. Take an initial position u = (0, y) $(0 \le y < 2\pi)$, and calculate the sequence $\{T^n u\}$ $(n \ge 1)$ and the rotation number $\nu(y)$. Using this information, determine the value y_0 to desired precision to give the KAM curve with ν . [Step 2] Calculate the mapping T Gauss and G' (x) with the following initial

[Step 2] Calculate the mapping T, G_{even} , and $G'_{\text{even}}(x)$ with the following initial conditions:

- (1) the position of phase space $(0, y_0)$.
- (2) the initial value of $G_{\text{even}}(x)$: $G_{\text{even}}(0) = y_0$.
- (3) the initial value of $G'_{\text{even}}(x)$: $G'_{\text{even}}(0) = 0$.

Appendix B

Using the following procedure and information regarding ζ_{-n} , we can numerically estimate the critical value a_c . We can use Procedure II when the initial condition is taken on the $x = \pi$ -axis.

Procedure II: Calculation of the critical value a_c .

[Step 1] Input $n \geq 2$.

[Step 2] Input a.

[Step 3] Take an initial position (0, y) $(0 \le y < 2\pi)$ on the y-axis (by symmetry, we need only consider the interval $0 \le y \le \pi$. See the last statement in §2.2.). Calculate the sequence $\{u_{-n} = T^{-n}u\}$ $(n \ge 1)$. Using Eq. (18) and the x-coordinate of u_{-n} , calculate $\zeta_{-1}, \dots, \zeta_{-n}$ in this order. If ζ_{-i} $(1 \le i \le n)$ does not satisfy the relation (37), we delete the initial position in the interval $(0, \pi)$ of initial conditions. If all ζ_{-i} $(1 \le i \le n)$ satisfy the relation (37), we keep such an initial position. We call this a "remaining interval".

[Step 4] Go back to [Step 3] and change the initial position. If the remaining interval exists after examination of the whole interval $(0, \pi)$, go back to [Step 2] and increase the value of a.

[Step 5] Repeating [Steps 2-4], if we find the values of a at which the remaining interval does not exist, we set $a_n = (\text{Minimum of } a)$.

[Step 6] Go back to [Step 1] and increase the value of n.

Appendix C

Let $\nu(y)$ be the calculated rotation number for the orbit with the initial condition (0, y). We can use Procedure III when the initial condition is taken on the $x = \pi$ -axis. **Procedure III:** Calculation of the critical value $a_c(\nu)$.

[Step 1] Input the rotation number ν .

[Step 2] Input a.

[Step 3] Take an initial position u = (0, y) $(0 \le y < 2\pi)$, and calculate the sequence $\{T^n u\}$ $(n \ge 1)$ and the rotation number $\nu(y)$. Find two initial positions, y_{\max} and y_{\min} , satisfying $\nu(y_{\max}) > \nu$ and $\nu(y_{\min}) < \nu$ with the condition $\Delta y = y_{\max} - y_{\min} < \epsilon$ (say, 10^{-7}).

[Step 4] Check the criterion (37) for the orbit starting from the initial condition $(0, y_i)$ $(y_i = y_{\min} + \Delta y \times i/10 : i = 0, \dots, 10)$. If no orbits satisfy Eq. (37), decrease the value of a and go back to [Step 3]. If this is not the case, increase a and go back to [Step 3]. Stop the procedure if the desired precision for a is obtained.

Appendix D

We can use Procedure IV when the initial condition is taken on the $x = \pi$ -axis. Preparation and stopping condition

The mapped points $\{x_n\}$ with $x_0 = 0$ and $G_{\text{even}}(x_n)$ are calculated and all data are sorted. The initial position $G_{\text{even}}(0)$ is determined by the rotation number. Increasing the total number of data from 2^{10} to 2^{17} , we check the convergence of calculated lengths. If the convergence is confirmed, we stop the doubling the number of data. If the convergence is not confirmed, the length is not listed in Table IV.

Note that we cannot calculate the length of $G_{\text{even}}^{(n)}(x)$ $(n \ge 2)$ if the length of $G_{\text{even}}'(x)$ is not calculated.

Procedure IV: Calculation of lengths.

[Step 1] Referring to the data $\{x_n\}$ and $\{G_{\text{even}}(x_n)\}$, calculate $G'_{\text{even}}(x_n)$ with $G'_{\text{even}}(0) = 0$. Sort the sequence $\{G'_{\text{even}}(x_n)\}$ according to the position x_n .

[Step 2] After interpolating the sorted data $\{G'_{\text{even}}(x_n)\}$, determine the second derivative. The spline interpolation is used.

[Step 3] Using a suitable initial condition for $G''_{\text{even}}(0)$, calculate the functional equation (34). Sort the data $\{G''_{\text{even}}(x_n)\}$.

[Step 4] Change $G''_{even}(0)$ in order that the difference between two series determined in [Steps 3 and 4] is in the acceptable error range. Here the method of regular false is used. The same method is used in [Step 8].

[Step 5] Calculate the functional equation for $G_{\text{even}}^{(3)}(x)$ with $G_{\text{even}}^{(3)}(0) = 0$. Here $G_{\text{even}}''(0)$ determined in [Step 4] is used. Sort the data $\{G_{\text{even}}^{(3)}(x_n)\}$.

[Step 6] After interpolating the sorted data $\{G_{\text{even}}^{(3)}(x_n)\}$, determine the 4-th derivative.

[Step 7] Using a suitable initial condition for $G_{\text{even}}^{(4)}(0)$, calculate the functional equation for $G_{\text{even}}^{(4)}(x)$. Sort the data $\{G_{\text{even}}^{(4)}(x_n)\}$.

[Step 8] Change $G_{\text{even}}^{(4)}(0)$ in order that the difference between two series determined in [Steps 7 and 8] is in the acceptable error range.

[Step 9] Calculate the functional equation for $G_{\text{even}}^{(5)}(x)$ with $G_{\text{even}}^{(5)}(0) = 0$. Here $G_{\text{even}}^{"}(0)$ and $G_{\text{even}}^{(4)}(0)$ determined in [Step 8] are used. Sort the data $\{G_{\text{even}}^{(5)}(x_n)\}$.

[Step 10] Calculate the lengths of $G_{\text{even}}^{(n)}(x)$ $(1 \le n \le 5)$.

References

- 1) G. B. Birkhoff, Acta Math. 43 (1920), 44.
- 2) M. R. Herman, Publ. Math. I. H. E. S. 49 (1979), 5.
- 3) J. Moser, SIAM REVIEW 28 (1986), 459.
- 4) S. Shinohara and Y. Aizawa, Prog. Theor. Phys. 97 (1997), 379.
 D. del-Castillo-Negrete, J. M. Green and P. J. Morrison, Physica D91 (1996), 1.
- 5) A. Denjoy, J. Math. Pures Appl. 11 (1932), 333.
 W. de Melo and S. van Strien, One-dimensional dynamics (Springer, 1993). See also 2).
- 6) Y. Katznelson and D. Ornstein, J. Analyse **60** (1989), 157.
- 7) S. Stirnemann, Commun. Math. Phys. **152** (1993), 369.
- 8) A. H. Osbaldestin and M. Y. Sarkis, J. of Phys. A20 (1987), L953.
- 9) B. R. Hunt, K. M. Khanin, Y. G. Sinai and J. A. Yorke, J. Stat. Phys. 85 (1996), 261.
- 10) L. E. Reichl, *The transition to chaos* (Springer-Verlag, 1992).
- R. S. Mackay, Renormalisation in area-preserving maps (World Scientific, 1993).
- R. De Vogelaere, in Contribution to the theory of nonlinear oscillation, Vol. IV, ed. S. Lefschetz (Princeton University Press, 1958), p. 53.
- 12) K. Tanikawa and Y. Yamaguchi, J. Math. Phys. 28 (1987), 921; 30 (1989), 608.
- Y. H. Ichikawa, T. Kamimura, T. Hatori and S. Y. Kim, Prog. Theor. Phys. Suppl. No. 98 (1989), 1.
- 14) J. N. Mather, Erg. Th. Dyn. Sys. 4 (1984), 301.
- 15) J. Greene, J. Math. Phys. 20 (1979), 1183.
- 16) R. S. Mackay and I. C. Percival, Commun. Math. Phys. 98 (1985), 469.
- 17) I. Jungreis, Erg. Th. Dyn. Sys. 11 (1991), 79.
- 18) S. Aubry, Physica **D7** (1983), 240.
- J. N. Mather, Publ. Math. I. H. E. S. **63** (1986), 153.
- 19) S. J. Shenker and L. P. Kadanoff, J. Stat. Phys. 27 (1982), 631.

- 20) A. Morbidelli and A. Giorgilli, J. Stat. Phys.
 ${\bf 78}$ (1994), 1607. 21) E. Lega and C. Froeschlé, Physica
 ${\bf D}$ ${\bf 95}$ (1996), 97.