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On the Schwinger Boson Representation of an Extended $(M+1)(N+1)$-Dimensional Algebra Containing the $su(M+1)$- and the $su(N,1)$-Algebra

Atsushi Kuriyama, João da Providência* and Masatoshi Yamamura

Faculty of Engineering, Kansai University, Suita 564-8680, Japan
**Departamento de Fisica, Universidade de Coimbra, P-3000 Coimbra, Portugal*

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With the aim of describing statistically mixed states in realistic many-body systems, the $su(n)$- and the $su(n,1)$-algebra in the Schwinger boson representation are presented in a form generalized from the $su(2)$- and the $su(1,1)$-algebra. The generators of the two algebras are expressed in terms of certain bilinear forms for boson operators. Some concrete examples are discussed in detail.

§1. Introduction

Since the $su(3)$-algebra was applied to a possible description of nuclear collective motion based on the shell model by Elliott,[1] the Lie algebra has played a central role, in some sense, in studies of nuclear collective motion. We can find some examples of studies based on the Lie algebra in Ref. 2). Much more detailed references until the beginning of 1990s are listed in the review article by Klein and Marshalek.[3] More recent information can be found in a special issue of J. of Phys. G.[4] Many of these works are restricted to compact algebras, for example, such as the $su(n)$- and the $so(n)$-algebra. Only a few studies related to non-compact algebras have been carried out, for example, by Ui[5] and Rowe.[6] This means that non-compact algebras seem for us to be not as important or not as interesting as compact algebras in the study of nuclear collective motion.

However, the simplest non-compact algebra, the $su(1,1)$-algebra, has an interesting physical feature. This algebra enables us to describe damped- and amplified-harmonic oscillations in the framework of quantum mechanics as a conservative system. This was pointed out by Vitiello and his coworkers.[7] A possible boson representation of the $su(1,1)$-algebra was presented by Schwinger, together with the $su(2)$-algebra.[8] This boson representation is formed of two kinds of boson operators. If we follow the scheme of Vitiello et al., one kind is introduced for the oscillating system, which we are concerned with, and the other for describing the mixture of pure states, in other words, the phase space doubling. With the use of the Schwinger representation, they investigated mainly the time-evolution of the vacuum. In addition to the vacuum, the time-evolutions of all other states have been investigated by the present authors.[9] The method adopted in Ref. 9) is that of the variational principle for the time-dependent Schrödinger equation, which was extensively studied in the TDHF theory in canonical form.[10] In this sense, we are able to refer to many-body systems described in terms of the $su(1,1)$-algebra in the

Schwinger boson representation as the $su(1,1)$-boson model. Furthermore, keeping the basic form of this model, we treated a boson system that is closely related to the $su(2)$-spin system,[11] for example, such as the superconducting state, and the results based on this treatment seem to be quite interesting. We mentioned that this model describes damped- and amplified-oscillations in terms of a conservative form. Then, the Hamiltonian governing the time-evolution of the system does not denote the total energy, but it plays a role similar to the Hamiltonian coming from the phase space doubling in thermo-field dynamics formalism.[12] However, we find an unsatisfactory point in this model.

As mentioned above the $su(1,1)$-boson model is composed of two degrees of freedom. One is the degree of freedom relevent to the oscillating system, in which we are interested. The other comes from the so-called phase-space doubling and plays a role in representing the statistical mixture of the pure states under investigation. Then, in the framework of this formalism, it may be impossible to treat dynamical motion, for example, such as damped- and amplified-oscillations, with explict use of the degrees of freedom representing the external environment, which may be in the background of this model. This is the unsatisfactory point. With the aim of removing this point, we propose an idea, and then, with the help of this idea, damped- and amplified-oscillations are described in terms of a system interacting with the external environment. The outline of this investigation is as follows: We describe the oscillating system and the external environment each in terms of a different kind of boson. Then, the total system is described in terms of two kinds of bosons, and this means that the Schwinger boson representation for the $su(2)$-algebra is useful. By taking into account successful aspects of the $su(1,1)$-boson model, we investigated the $su(1,1)$-spin-like behavior in the $su(2)$-spin system. However, in the framework of the $su(2)$-boson model, we can describe only the pure state. Then, with the aim of describing the statistically mixed state, we extended this method to the case of the $su(2,1)$-boson model consisting of three kinds of bosons.[14),15] The first and the second kind of boson play the same role as those in the $su(2)$-boson model. The third plays the role of the phase space doubling in the oscillating system under investigation. On the basis of this model, we investigated the statistically mixed state with some interesting results.

The main aim of this paper is to present a possible general form of the $su(n)$- and the $su(n,1)$-algebra in the Schwinger boson representation. As mentioned above, the $su(2,1)$-boson model given in Refs. 14) and 15) has quite interesting features, but, still it is a toy model. If we intend to apply the basic idea in Ref. 15) to realistic many-body systems, first we must construct, inevitably, the $su(n)$- and the $su(n,1)$-algebra in the Schwinger boson representation as a possible receptacle. For this purpose, we prepare $(M+1)(N+1)$ kinds of boson operators, where N and M denote arbitrary positive integers. Then, it may be possible to construct the $su(M+1)$- and the $su(N,1)$-algebra by constructing bilinear forms for the boson operators. Clearly, the case ($M=1$, $N=0$) and the case ($M=0$, $N=1$) are, respectively, reduced to the $su(2)$- and the $su(1,1)$-algebra. In the present paper, after a general form is given, some typical examples are discussed.

In §§2 and 3, all the generators of the $su(M+1)$- and the $su(N,1)$-algebra in

terms of bilinear forms for the boson operators and their linearly-independent basis are given in general form. In §4, three simple examples are presented: the case (M arbitrary, $N = 0$), the case ($M = 0$, N arbitrary), and the case ($M = 1$, $N = 1$). Section 5 is devoted to treating the cases ($M = 2$, $N = 1$) and ($M = 1$, $N = 2$). With these cases, we can treat the $su(3)$- and the $su(2,1)$-algebra in terms of six kinds of bosons. In §6, some points are discussed, and we give concluding remarks.

§2. Bilinear forms expressed in terms of boson operators

Let us investigate a boson space which is constructed with $(M+1)(N+1)$ kinds of boson operators. We denote them $(\hat{a}_i, \hat{a}_i^*)$, $(\hat{a}^p, \hat{a}^{p*})$, $(\hat{b}_i^p, \hat{b}_i^{p*})$ and $(\hat{b}, \hat{b}^*)$: $i = 1, 2, \cdots, M$ and $p = 1, 2, \cdots, N$. They obey the commutation relations

$$\begin{aligned} &[\hat{a}_i, \hat{a}_j^*] = \delta_{ij}, \quad [\hat{a}^p, \hat{a}^{q*}] = \delta^{pq}, \\ &[\hat{b}_i^p, \hat{b}_j^{q*}] = \delta_{ij}\delta^{pq}, \quad [\hat{b}, \hat{b}^*] = 1, \\ &[\text{other combinations}] = 0. \end{aligned} \tag{2·1}$$

In the above-described boson space, let us introduce the following bilinear forms expressed in terms of the above boson operators:

$$\hat{E}^i = \hat{a}_i^* \hat{b} + \sum_{p=1}^{N} \hat{a}^{p*} \hat{b}_i^p,$$

$$\hat{E}_i = \hat{b}^* \hat{a}_i + \sum_{p=1}^{N} \hat{b}_i^{p*} \hat{a}^p, \tag{2·2a}$$

$$\hat{E}_i^j = \hat{a}_i^* \hat{a}_j - \sum_{p=1}^{N} \hat{b}_j^{p*} \hat{b}_i^p, \tag{2·2b}$$

$$\hat{E} = \sum_{p=1}^{N} \hat{a}^{p*} \hat{a}^p - \hat{b}^* \hat{b}. \tag{2·2c}$$

It can be proved that these bilinear forms obey the $u(M+1)$-algebra:

$$\hat{E}_i^* = \hat{E}^i, \quad \hat{E}_j^{i*} = \hat{E}_i^j, \quad \hat{E}^* = \hat{E}, \tag{2·3}$$

$$[\hat{E}^i, \hat{E}^j] = 0, \quad [\hat{E}^i, \hat{E}_j] = +\hat{E}_i^j + \delta_{ij}\hat{E}, \tag{2·4a}$$

$$[\hat{E}_i^j, \hat{E}^k] = \delta_{jk}\hat{E}^i, \quad [\hat{E}, \hat{E}^k] = \hat{E}^k, \tag{2·4b}$$

$$[\hat{E}_i^j, \hat{E}_k^l] = \delta_{jk}\hat{E}_i^l - \delta_{il}\hat{E}_k^j, \quad [\hat{E}_i^j, \hat{E}] = 0. \tag{2·4c}$$

We can construct the $su(M+1)$-algebra by defining the generators $\{\hat{S}^i, \hat{S}_i, \hat{S}_i^j\}$ in the form

$$\hat{S}^i = \hat{E}^i, \quad \hat{S}_i = \hat{E}_i, \quad \hat{S}_i^j = \hat{E}_i^j + \delta_{ij}\hat{E}. \tag{2·5}$$

In association with the set $\{\hat{E}^i, \hat{E}_i, \hat{E}_i^j, \hat{E}\}$, we introduce the following bilinear forms:

$$\hat{F}^p = \hat{b}^* \hat{a}^{p*} - \sum_{i=1}^{M} \hat{b}_i^{p*} \hat{a}_i^*,$$

$$\hat{F}_p = \hat{a}^p\hat{b} - \sum_{i=1}^{M} \hat{a}_i\hat{b}_i^p, \tag{2·6a}$$

$$\hat{F}_q^p = \sum_{i=1}^{M} \hat{b}_i^q\hat{b}_i^{p*} + \hat{a}^{p*}\hat{a}^q, \tag{2·6b}$$

$$\hat{F} = \hat{b}\hat{b}^* + \sum_{i=1}^{M} \hat{a}_i^*\hat{a}_i. \tag{2·6c}$$

It can be proved that they obey the $u(N,1)$-algebra:

$$\hat{F}_p^* = \hat{F}^p, \quad \hat{F}_p^{q*} = \hat{F}_q^p, \quad \hat{F}^* = \hat{F}, \tag{2·7}$$

$$[\hat{F}^p, \hat{F}^q] = 0, \quad [\hat{F}^p, \hat{F}_q] = -\delta^{pq}\hat{F} - \hat{F}_q^p, \tag{2·8a}$$

$$[\hat{F}_q^p, \hat{F}^r] = \delta^{qr}\hat{F}^p, \quad [\hat{F}, \hat{F}^r] = \hat{F}^r, \tag{2·8b}$$

$$[\hat{F}_q^p, \hat{F}_s^r] = \delta^{qr}\hat{F}_s^p - \delta^{ps}\hat{F}_q^r, \quad [\hat{F}_q^p, \hat{F}] = 0. \tag{2·8c}$$

We are also able to construct the $su(N,1)$-algebra by defining the generators $\{\hat{T}^p, \hat{T}_p, \hat{T}_q^p\}$ in the forms

$$\hat{T}^p = \hat{F}^p, \quad \hat{T}_p = \hat{F}_p, \quad \hat{T}_q^p = \delta^{pq}\hat{F} + \hat{F}_q^p. \tag{2·9}$$

For the bilinear forms given by the relations (2·2) and (2·6), it should be noted that any generator in the set $\{\hat{E}^i, \hat{E}_i, \hat{E}_i^j, \hat{E}\}$ commutes with any generator in the set $\{\hat{F}^p, \hat{F}_p, \hat{F}_p^q, \hat{F}\}$:

$$[\hat{E}^i, \hat{F}^p] = 0, \text{ etc.} \tag{2·10a}$$

We can also prove the relations

$$[\hat{S}^i, \hat{T}^p] = 0, \text{ etc.} \tag{2·10b}$$

Further, it is possible to define an operator $\hat{S}$ that commutes with the generators of the $su(M+1)$-algebra:

$$\hat{S} = \sum_{i=1}^{M} \hat{E}_i^i - \hat{E}. \tag{2·11a}$$

In the case of the $su(N,1)$-algebra, we have

$$\hat{T} = \hat{F} - \sum_{p=1}^{N} \hat{F}_p^p. \tag{2·11b}$$

However, $\hat{T}$ is essentially identical to $\hat{S}$:

$$\hat{T} = \hat{S} - MN + 1. \tag{2·11c}$$

The case ($M = 1$, $N = 0$) corresponds to ($i = 1$, $p = 0$), and we can write down $\hat{S}^i$, $\hat{S}_i$, $\hat{S}_i^j$ and $\hat{S}$ as

$$\hat{S}^1 = \hat{a}^*\hat{b}, \quad \hat{S}_1 = \hat{b}^*\hat{a}, \quad \hat{S}_1^1 = \hat{a}^*\hat{a} - \hat{b}^*\hat{b}, \tag{2·12a}$$

$$\hat{S} = \hat{a}^*\hat{a} + \hat{b}^*\hat{b}. \tag{2·12b}$$

Here, omitting the index $i = 1$, we denote $\hat{a}_1$ as $\hat{a}$. If the notation is changed according to $\hat{S}^1 = \hat{\sigma}_+$, $\hat{S}_1 = \hat{\sigma}_-$, $\hat{S}^1_1 = 2\hat{\sigma}_0$ and $\hat{S} = 2\hat{\sigma}$, the set $\{\sigma_{\pm,0}\}$ forms the $su(2)$-algebra in the Schwinger representation, and $\hat{\sigma}$ denotes the magnitude of the $su(2)$-spin.[8] In the case ($M = 0$, $N = 1$), i.e. ($i = 0$, $p = 1$), denoting $\hat{a}^1$ as $\hat{a}$, we have the following form for $\hat{T}^1$, $\hat{T}_1$ and $\hat{T}^1_1$:

$$\hat{T}^1 = \hat{b}^*\hat{a}^*, \quad \hat{T}_1 = \hat{a}\hat{b}, \quad \hat{T}^1_1 = \hat{b}\hat{b}^* + \hat{a}^*\hat{a}, \tag{2·13a}$$

$$\hat{T} = \hat{b}\hat{b}^* - \hat{a}^*\hat{a}. \tag{2·13b}$$

In this case, the change of the notation according to $\hat{T}^1 = \hat{\tau}_+$, $\hat{T}_1 = \hat{\tau}_-$, $\hat{T}^1_1 = 2\hat{\tau}_0$ and $\hat{T} = 2\hat{\tau}$ shows us that the set $\{\hat{\tau}_{\pm,0}\}$ forms the $su(1,1)$-algebra in the Schwinger boson representation, and $\hat{\tau}$ denotes the magnitude of the $su(1,1)$-spin.[8] The above results indicate that it may be permitted to refer to the forms (2·5) and (2·9) as the Schwinger boson representations for the generators of the $su(M+1)$- and the $su(N,1)$-algebra, respectively.

The Casimir operators of the $u(M+1)$- and the $u(N,1)$-algebra are defined, respectively, as

$$\begin{aligned}
\hat{\Gamma}_{u(M+1)} &= +\sum_{i=1}^{M}(\hat{E}^i\hat{E}_i + \hat{E}_i\hat{E}^i) + \sum_{i,j=1}^{M}\hat{E}^i_j\hat{E}^j_i + \hat{E}^2 \\
&= +2\sum_{i=1}^{M}\hat{E}^i\hat{E}_i + \sum_{i\neq j}\hat{E}^i_j\hat{E}^j_i \\
&\quad + \sum_{i=1}^{M}\hat{E}^i_i(\hat{E}^i_i - 1) + \hat{E}(\hat{E} - M),
\end{aligned} \tag{2·14a}$$

$$\begin{aligned}
\hat{\Gamma}_{u(N,1)} &= -\sum_{p=1}^{N}(\hat{F}^p\hat{F}_p + \hat{F}_p\hat{F}^p) + \sum_{p,q=1}^{N}\hat{F}^p_q\hat{F}^q_p + \hat{F}^2 \\
&= -2\sum_{p=1}^{N}\hat{F}^p\hat{F}_p + \sum_{p\neq q}\hat{F}^q_p\hat{F}^p_q \\
&\quad + \sum_{p=1}^{N}\hat{F}^p_p(\hat{F}^p_p - 1) + \hat{F}(\hat{F} - N).
\end{aligned} \tag{2·14b}$$

Here, we have restricted ourselves to treat the Casimir operators which are of bilinear form with respect to the generators, and hereafter we treat only the bilinear Casimir operators. We are also able to construct the Casimir operators for the $su(M+1)$- and the $su(N,1)$-algebra. They are given in the following forms:

$$\begin{aligned}
\hat{\Gamma}_{su(M+1)} &= +\sum_{i=1}^{M}(\hat{S}^i\hat{S}_i + \hat{S}_i\hat{S}^i) + \sum_{i,j=1}^{M}\hat{S}^i_j\hat{S}^j_i - (M+1)^{-1}\left(\sum_{i=1}^{M}\hat{S}^i_i\right)^2 \\
&= \hat{\Gamma}_{u(M+1)} - (M+1)^{-1}\left(\hat{E} - \sum_{i=1}^{M}\hat{E}^i_i\right)^2,
\end{aligned} \tag{2·15a}$$

$$\begin{aligned}\hat{\Gamma}_{su(N,1)} &= -\sum_{p=1}^{N}(\hat{T}^p\hat{T}_p+\hat{T}_p\hat{T}^p)+\sum_{p,q=1}^{N}\hat{T}^p_q\hat{T}^q_p-(N+1)^{-1}\left(\sum_{p=1}^{N}\hat{T}^p_p\right)^2\\ &= \hat{\Gamma}_{u(N,1)}-(N+1)^{-1}\left(\hat{F}-\sum_{p=1}^{N}\hat{F}^p_p\right)^2. \qquad (2\cdot15\text{b})\end{aligned}$$

All the generators for each algebra commute with their corresponding Casimir operator. For the sake of the relation (2·10), all the generators also commute with the four Casimir operators.

§3. Construction of the linearly independent basis

For studies of the algebra, it is self-evident that the construction of the orthogonal basis is indispensable. This task is performed by solving the eigenvalue problem for mutually commutable hermite operators. In the present case, the mutually commuting operators are, for example, $\hat{E}^i_i$ $(i=1,2,\cdots,M)$, $\hat{E}$, $\hat{F}^p_p$ $(p=1,2,\cdots,N)$, $\hat{F}$, $\hat{\Gamma}_{u(M+1)}$ and $\hat{\Gamma}_{u(N,1)}$. In this paper, mainly, the eigenvalue problem for the above operators is investigated for the purpose of obtaining the linearly independent basis. With the use of this basis, we are able to obtain an orthogonal basis.

First, we note that the operators $\hat{E}^i$ and $\hat{F}^p$ play the role of the raising operators from the state with the minimum weight, which we denote $|\nu_b\rangle$. The state $|\nu_b\rangle$ is defined as the state obeying

$$\hat{E}_i|\nu_b\rangle=\hat{F}_p|\nu_b\rangle=0. \qquad (3\cdot1)$$

As is clear from the forms (2·2a) and (2·6a), the state $|\nu_b\rangle$ does not contain any of $\hat{a}^*_i$ and $\hat{a}^{p*}$. In other words, the state $|\nu_b\rangle$ is constructed from the bosons $\hat{b}^{p*}_i$ and $\hat{b}^*$. Then, we introduce the following set of states:

$$|n_1\cdots n_N,m_1\cdots m_M;\nu_b\rangle=(\hat{F}^1)^{n_1}\cdots(\hat{F}^N)^{n_N}\cdot(\hat{E}^1)^{m_1}\cdots(\hat{E}^M)^{m_M}|\nu_b\rangle, \qquad (3\cdot2)$$

$$n_1,\cdots,n_N,m_1,\cdots,m_M=0,1,2,\cdots. \qquad (3\cdot3)$$

Hereafter, we adopt abbreviated forms for $n_1,\cdots,n_N$ and $m_1,\cdots,m_M$ and the product of $\hat{F}^1,\cdots,\hat{F}^N$, $\hat{E}^1,\cdots\hat{E}^M$ as follows:

$$\begin{aligned}\boldsymbol{n}&=(n_1\cdots n_N), \quad \hat{F}(\boldsymbol{n})=(\hat{F}^1)^{n_1}\cdots(\hat{F}^N)^{n_N},\\ \boldsymbol{m}&=(m_1\cdots m_M), \quad \hat{E}(\boldsymbol{m})=(\hat{E}^1)^{m_1}\cdots(\hat{E}^M)^{m_M}. \qquad (3\cdot4)\end{aligned}$$

Then, the state (3·2) can be abbreviated as

$$|\boldsymbol{n},\boldsymbol{m};\nu_b\rangle=\hat{F}(\boldsymbol{n})\hat{E}(\boldsymbol{m})|\nu_b\rangle. \qquad (3\cdot5)$$

In the present paper, the normalization constant will not be given explicitly. We are able to prove that the state (3·5) satisfies the relations

$$\hat{E}^i_i|\boldsymbol{n},\boldsymbol{m};\nu_b\rangle = m_i|\boldsymbol{n},\boldsymbol{m};\nu_b\rangle+\hat{F}(\boldsymbol{n})\hat{E}(\boldsymbol{m})\cdot\hat{E}^i_i|\nu_b\rangle, \qquad (3\cdot6\text{a})$$

$$\begin{aligned}
\hat{E}|\boldsymbol{n},\boldsymbol{m};\nu_b\rangle &= m|\boldsymbol{n},\boldsymbol{m};\nu_b\rangle + \hat{F}(\boldsymbol{n})\hat{E}(\boldsymbol{m})\cdot\hat{E}|\nu_b\rangle, \\
m &= \sum_{i=1}^{M} m_i, && (3\cdot 6\mathrm{b}) \\
\hat{F}^p_p|\boldsymbol{n},\boldsymbol{m};\nu_b\rangle &= n_p|\boldsymbol{n},\boldsymbol{m};\nu_b\rangle + \hat{F}(\boldsymbol{n})\hat{E}(\boldsymbol{m})\cdot\hat{F}^p_p|\nu_b\rangle, && (3\cdot 7\mathrm{a}) \\
\hat{F}|\boldsymbol{n},\boldsymbol{m};\nu_b\rangle &= n|\boldsymbol{n},\boldsymbol{m};\nu_b\rangle + \hat{F}(\boldsymbol{n})\hat{E}(\boldsymbol{m})\cdot\hat{F}|\nu_b\rangle, \\
n &= \sum_{p=1}^{N} n_p, && (3\cdot 7\mathrm{b}) \\
\hat{\Gamma}_{u(M+1)}|\boldsymbol{n},\boldsymbol{m};\nu_b\rangle &= \hat{F}(\boldsymbol{n})\hat{E}(\boldsymbol{m})\cdot\hat{\Gamma}_{u(M+1)}|\nu_b\rangle, && (3\cdot 8\mathrm{a}) \\
\hat{\Gamma}_{u(N,1)}|\boldsymbol{n},\boldsymbol{m};\nu_b\rangle &= \hat{F}(\boldsymbol{n})\hat{E}(\boldsymbol{m})\cdot\hat{\Gamma}_{u(N,1)}|\nu_b\rangle. && (3\cdot 8\mathrm{b})
\end{aligned}$$

Let the state $|\nu_b\rangle$ be expressed as an eigenstate of $\hat{E}^i_i$, $\hat{E}$, $\hat{F}^p_p$, $\hat{F}$, $\hat{\Gamma}_{u(M+1)}$ and $\hat{\Gamma}_{u(N,1)}$. Then, $|\boldsymbol{n},\boldsymbol{m};\nu_b\rangle$ can be regarded as the eigenstate of these operators. The next task is to investigate the state $|\nu_b\rangle$.

For the construction of $|\nu_b\rangle$, first, we introduce the operator

$$\hat{D}^{ip} = \hat{b}^{p*}_i\hat{b} + \hat{a}^{p*}\hat{a}_i. \quad (i = 1, 2, \cdots, M, \quad p = 1, 2, \cdots, N) \tag{3·9}$$

We now list some important properties, which will be used on several occasions:

$$\begin{aligned}
&[\hat{D}^{ip}, \hat{D}^{jq}] = 0, && (3\cdot 10\mathrm{a}) \\
&[\hat{E}^k_l, \hat{D}^{ip}] = -\delta_{il}\hat{D}^{kp}, \quad [\hat{E}, \hat{D}^{ip}] = +\hat{D}^{ip}, && (3\cdot 10\mathrm{b}) \\
&[\hat{F}^s_r, \hat{D}^{ip}] = +\delta^{pr}\hat{D}^{is}, \quad [\hat{F}, \hat{D}^{ip}] = -\hat{D}^{ip}, && (3\cdot 10\mathrm{c}) \\
&[\hat{E}^j, \hat{D}^{ip}] = 0, \quad [\hat{E}_j, \hat{D}^{ip}] = -\hat{b}^{p*}_i\hat{a}_j + \hat{b}^{p*}_j\hat{a}_i, && (3\cdot 10\mathrm{d}) \\
&[\hat{F}_q, \hat{D}^{ip}] = 0, \quad [\hat{F}^q, \hat{D}^{ip}] = +\hat{a}^{p*}\hat{b}^q_i - \hat{a}^{q*}\hat{b}^p_i. && (3\cdot 10\mathrm{e})
\end{aligned}$$

With use of the above-defined operator $\hat{D}^{ip}$, we introduce the state

$$|\boldsymbol{l};\nu\rangle = \hat{D}(\boldsymbol{l})|\nu\rangle, \tag{3·11}$$

$$|\nu\rangle = (\hat{b}^*)^\nu|0\rangle. \quad (\nu = 0, 1, 2, \cdots) \tag{3·12}$$

Here, $\boldsymbol{l}$ and $\hat{D}(\boldsymbol{l})$ are given by

$$\begin{aligned}
&\boldsymbol{l} = (l_{11}\cdots l_{M1}, l_{12}\cdots l_{M2}, \cdots, l_{1N}\cdots l_{NM}), \\
&\hat{D}(\boldsymbol{l}) = (\hat{D}^{11})^{l_{11}}\cdots(\hat{D}^{M1})^{l_{M1}}\cdots(\hat{D}^{1N})^{l_{1N}}\cdots(\hat{D}^{MN})^{l_{MN}}.
\end{aligned} \tag{3·13}$$

Here, $|0\rangle$ denotes the vacuum of all the bosons in the present formalism. The quantities l_{11}, etc., take non-negative integer values:

$$l_{11} = 0, 1, 2, \cdots, \text{ etc.} \tag{3·14}$$

The operator $\hat{D}^{ip}$ contains $(\hat{a}, \hat{a}^*)$-type bosons in the form $\hat{a}^*\hat{a}$, and it operates on the state $|\nu\rangle$ containing only a $\hat{b}^*$-boson. Therefore, we can see that the set of states $\{|\boldsymbol{l};\nu\rangle\}$ is orthogonal:

$$\begin{aligned}
|\boldsymbol{l};\nu\rangle = &(\hat{b}^{1*}_1)^{l_{11}}\cdots(\hat{b}^{1*}_M)^{l_{M1}}\cdots \\
&\cdot(\hat{b}^{N*}_1)^{l_{1N}}\cdots(\hat{b}^{N*}_M)^{l_{MN}}\cdot(\hat{b}^*)^{\nu-(l_{11}+\cdots+l_{MN})}|0\rangle.
\end{aligned} \tag{3·15}$$

Further, $|\boldsymbol{l};\nu\rangle$ obeys

$$\hat{E}_i|\boldsymbol{l};\nu\rangle = \hat{F}_p|\boldsymbol{l};\nu\rangle = 0. \tag{3·16}$$

We can show that the state $|\boldsymbol{l};\nu\rangle$ is an eigenstate of $\hat{E}_i^i$, $\hat{E}$, $\hat{F}_p^p$ and $\hat{F}$:

$$\hat{E}_i^i|\boldsymbol{l};\nu\rangle = -l_i|\boldsymbol{l};\nu\rangle, \tag{3·17a}$$
$$\hat{E}|\boldsymbol{l};\nu\rangle = (l-\nu)|\boldsymbol{l};\nu\rangle, \tag{3·17b}$$
$$\hat{F}_p^p|\boldsymbol{l};\nu\rangle = (l^p+M)|\boldsymbol{l};\nu\rangle, \tag{3·18a}$$
$$\hat{F}|\boldsymbol{l};\nu\rangle = (\nu-l+1)|\boldsymbol{l};\nu\rangle. \tag{3·18b}$$

Here, l_i, l^p and l are defined as

$$l_i = \sum_{p=1}^{N} l_{ip}, \quad l^p = \sum_{i=1}^{M} l_{ip}, \quad l = \sum_{i=1}^{M}\sum_{p=1}^{N} l_{ip}. \tag{3·19}$$

As shown in Eqs. (3·16)–(3·18), $|\boldsymbol{l};\nu\rangle$ is an eigenstate of $\hat{E}_i^i$, $\hat{E}$, $\hat{F}_p^p$ and $\hat{F}$ and obeys the condition (3·1). However, in general, $|l;\nu\rangle$ is not an eigenstate of $\hat{\Gamma}_{u(M+1)}$ and $\hat{\Gamma}_{u(N,1)}$. Then, $|\nu_b\rangle$ can be obtained by taking linear combinations of the orthogonal set $\{|\boldsymbol{l};\nu\rangle\}$ under the conditions (3·17a) and (3·18a):

$$|\nu_b\rangle = |\boldsymbol{k};\nu\rangle = \sum_{\boldsymbol{l}} C(\boldsymbol{k},\boldsymbol{l})|\boldsymbol{l};\nu\rangle, \tag{3·20}$$

$$\hat{E}_i^i|\boldsymbol{k};\nu\rangle = -l_i|\boldsymbol{k};\nu\rangle,$$
$$\hat{E}|\boldsymbol{k};\nu\rangle = (l-\nu)|\boldsymbol{k};\nu\rangle, \tag{3·21a}$$
$$\hat{F}_p^p|\boldsymbol{k};\nu\rangle = (l^p+M)|\boldsymbol{k};\nu\rangle,$$
$$\hat{F}|\boldsymbol{k};\nu\rangle = (\nu-l+1)|\boldsymbol{k};\nu\rangle, \tag{3·21b}$$
$$\hat{\Gamma}_{u(M+1)}|\boldsymbol{k};\nu\rangle = \gamma_{u(M+1)}(\boldsymbol{k})|\boldsymbol{k};\nu\rangle,$$
$$\hat{\Gamma}_{u(N,1)}|\boldsymbol{k};\nu\rangle = \gamma_{u(N,1)}(\boldsymbol{k})|\boldsymbol{k};\nu\rangle. \tag{3·21c}$$

Here, the quantity $\boldsymbol{k}$ is used for specifying the eigenvalues of the operators $\hat{\Gamma}_{u(M+1)}$ and $\hat{\Gamma}_{u(N,1)}$, i.e., $\gamma_{u(M+1)}(\boldsymbol{k})$ and $\gamma_{u(N,1)}(\boldsymbol{k})$.

Thus, we are able to obtain the eigenstate in the form

$$|\boldsymbol{n},\boldsymbol{m};\nu_b\rangle = |\boldsymbol{n},\boldsymbol{m},\boldsymbol{k};\nu\rangle = \sum_{\boldsymbol{l}} C(\boldsymbol{k},\boldsymbol{l})\hat{F}(\boldsymbol{n})\hat{E}(\boldsymbol{m})\hat{D}(\boldsymbol{l})|\nu\rangle. \tag{3·22}$$

The eigenvalues of $\hat{E}_i^i$, $\hat{E}$, $\hat{F}_p^p$, $\hat{F}$, $\hat{\Gamma}_{u(M+1)}$ and $\hat{\Gamma}_{u(N,1)}$ are given as (m_i-l_i), $(m+l-\nu)$, (n_p+l^p+M), $(n+\nu-l+1)$, $\gamma_{su(M+1)}(\boldsymbol{k})$ and $\gamma_{u(N,1)}(\boldsymbol{k})$. However, it should be noted that, in general, the set $\{|\boldsymbol{n},\boldsymbol{m},\boldsymbol{k};\nu\rangle\}$ forms a linearly independent basis, but does not always form an orthogonal basis. The number of mutually commuting hermitian operators is less than that of the quantum numbers specifying the orthogonal basis. The state $|\boldsymbol{n},\boldsymbol{m},\boldsymbol{k};\nu\rangle$ is also an eigenstate of $\hat{S}_i^i$, $\hat{S}$, $\hat{T}_p^p$, $\hat{T}$, $\hat{\Gamma}_{su(M+1)}$ and $\hat{\Gamma}_{su(N,1)}$.

The eigenvalues which we denote s_i^i, s, t_p^p, t, $\gamma_{su(M+1)}$ and $\gamma_{su(N,1)}$ are given by

$$s_i^i = m_i - l_i + m + l - \nu, \quad s = \nu - 2l, \tag{3·23a}$$

$$t_p^p = n_p + l^p + n + \nu - l + M + 1, \quad t = \nu - 2l - MN + 1, \tag{3·23b}$$

$$\gamma_{su(M+1)} = \gamma_{u(M+1)}(\boldsymbol{k}) - (M+1)^{-1}(2l-\nu)^2,$$

$$\gamma_{su(N,1)} = \gamma_{u(N,1)}(\boldsymbol{k}) - (N+1)^{-1}(2l-\nu+MN-1)^2. \tag{3·23c}$$

From the following argument, we can determine a condition under which the state $|\boldsymbol{l};\nu\rangle$ given in Eq. (3·15) is directly an eigenstate of $\hat{\Gamma}_{u(M+1)}$ and $\hat{\Gamma}_{u(N,1)}$. The numbers of l_{ip} and (l_i, l^p) are MN and $(M+N)$, respectively. Therefore, if we have the inequality $MN \leq M+N$, it is unnecessary to solve the eigenvalue equations (3·21c). This condition is realized in the cases (M arbitrary, $N=0$ or 1), ($M=0$ or 1, N arbitrary) and ($M=2$, $N=2$).

In this paper, detailed results are given only in the cases satisfying the condition $MN \leq M+N$. The condition $MN > M+N$ will be discussed in detail in a forthcoming paper.

§4. Three simple examples

In the final part of §2, we mentioned that the cases ($M=1$, $N=0$) and ($M=0$, $N=1$) correspond to the $su(2)$- and the $su(1,1)$-algebra in the Schwinger boson representation, respectively. In this section, we consider three other simple cases.

4.1. *The case* (M *arbitrary,* $N=0$): *the* $su(M+1)$*-algebra*

This case corresponds to ($i=1,\cdots,M$, $p=0$) and is constructed in terms of the boson operators $(\hat{a}_i, \hat{a}_i^*; i=1,2,\cdots,M)$ and $(\hat{b}, \hat{b}^*)$. Then, $\hat{S}^i$, $\hat{S}_i$, $\hat{S}_i^j$ and S are reduced to the following forms:

$$\hat{S}^i = \hat{a}_i^*\hat{b}, \quad \hat{S}_i = \hat{b}^*\hat{a}_i, \quad \hat{S}_i^j = \hat{a}_i^*\hat{a}_j - \delta_{ij}\hat{b}^*\hat{b}, \tag{4·1a}$$

$$\hat{S} = \sum_{i=1}^{M} \hat{a}_i^*\hat{a}_i + \hat{b}^*\hat{b}. \tag{4·1b}$$

No operator in the set $\{\hat{T}^p, \hat{T}_p, \hat{T}_q^p\}$ and $\hat{D}^{ip}$ is related to this case. In the state (3·22), $\boldsymbol{n} = \boldsymbol{l} = 0$ and, then, $\boldsymbol{k} = 0$ and $C(\boldsymbol{k}=0, \boldsymbol{l}=0) = 1$. Therefore, we have

$$\begin{aligned} |0, \boldsymbol{m}, 0; \nu\rangle &= \hat{E}(\boldsymbol{m})|\nu\rangle = (\hat{E}^1)^{m_1}\cdots(\hat{E}^M)^{m_M}|\nu\rangle \\ &= (\hat{S}^1)^{m_1}\cdots(\hat{S}^M)^{m_M}\cdot(\hat{b}^*)^{\nu}|0\rangle \\ &= (\hat{a}_1^*)^{m_1}\cdots(\hat{a}_M^*)^{m_M} \\ &\quad \cdot(\hat{b}^*)^{\nu-(m_1+\cdots+m_M)}|0\rangle. \end{aligned} \tag{4·2}$$

In this case, $\{|0, \boldsymbol{m}, 0; \nu\rangle\}$ is an orthogonal set. The state (4·2) is the eigenstate of the operator $\hat{\Gamma}_{u(M+1)}$, and the eigenvalues of the operators $\hat{E}_i^i$, $\hat{E}$ and $\hat{\Gamma}_{u(M+1)}$ are given as m_i, $(m-\nu)$ and $\nu(\nu+M)$, respectively. Therefore, the eigenvalues of $\hat{S}_i^i$, $\hat{S}$ and $\hat{\Gamma}_{su(M+1)}$ are expressed as $(m_i+m-\nu)$, ν and $M(M+1)[\nu/(M+1)]\cdot[\nu/(M+1)+1]$,

respectively. This case is closely related with the Holstein-Primakoff boson expansion in the symmetric representation of the $su(M+1)$-algebra:[3)]

$$\hat{S}^i = \hat{a}_i^* \cdot \sqrt{\nu - \sum_{k=1}^{M} \hat{a}_k^* \hat{a}_k}, \quad \hat{S}_i = \sqrt{\nu - \sum_{k=1}^{M} \hat{a}_k^* \hat{a}_k} \cdot \hat{a}_i,$$
$$\hat{S}_i^j = \hat{a}_i^* \hat{a}_j - \delta_{ij} \left(\nu - \sum_{k=1}^{M} \hat{a}_k^* \hat{a}_k \right). \tag{4·3}$$

The case $M = 1$ is reduced to the well-known Holstein-Primakoff boson expansion of the $su(2)$-algebra.

4.2. *The case* ($M = 0$, N *arbitrary*): *the* $su(N,1)$*-algebra*

This case corresponds to $(i = 0, p = 1, \cdots, N)$ and is formed by the boson operators $(\hat{a}^p, \hat{a}^{p*}; p = 1, \cdots, N)$ and $(\hat{b}, \hat{b}^*)$. Then, the generators of the present algebra $\hat{T}^p$, $\hat{T}_p$ and $\hat{T}_q^p$ are reduced to the following forms:

$$\hat{T}^p = \hat{b}^* \hat{a}^{p*}, \quad \hat{T}_p = \hat{a}^p \hat{b}, \quad \hat{T}_q^p = \delta^{pq} \hat{b}\hat{b}^* + \hat{a}^{p*}\hat{a}^q, \tag{4·4a}$$

$$\hat{T} = \hat{b}\hat{b}^* - \sum_{p=1}^{N} \hat{a}^{p*}\hat{a}^p. \tag{4·4b}$$

None of $\hat{S}^i$, $\hat{S}_i$, $\hat{S}_j^i$ and $\hat{D}^{ip}$ is related to the present case. In the state (3·23), $\boldsymbol{m} = \boldsymbol{l} = 0$ and, then, $\boldsymbol{k} = 0$ and $C(\boldsymbol{k} = 0, \boldsymbol{l} = 0) = 1$. Therefore, we have the following state:

$$\begin{aligned} |\boldsymbol{n}, 0, 0; \nu\rangle &= \hat{F}(\boldsymbol{n})|\nu\rangle = (\hat{F}^1)^{n1} \cdots (\hat{F}^N)^{n_N} |\nu\rangle \\ &= (\hat{T}^1)^{n_1} \cdots (\hat{T}^N)^{n_N} \cdot (\hat{b}^*)^\nu |0\rangle \\ &= (\hat{a}^{1*})^{n_1} \cdots (\hat{a}^{N*})^{n_N} \cdot (\hat{b}^*)^{\nu + (n_1 + \cdots + n_N)} |0\rangle. \end{aligned} \tag{4·5}$$

It may be self-evident that the set $\{|\boldsymbol{n}, 0, 0; \nu\rangle\}$ is orthgonal. In this case, also, the state (4·5) is directly an eigenstate of $\hat{\Gamma}_{u(N,1)}$, and it is not necessary to solve the eigenvalue equation (3·21c). The eigenvalues of $\hat{F}_p^p$, $\hat{F}$ and $\hat{\Gamma}_{u(N,1)}$ are given as n_p, $(n + \nu + 1)$ and $(\nu + 1)(\nu + 1 - N)$, respectively. Then, the eigenvalues of $\hat{T}_p^p$, $\hat{T}$ and $\hat{\Gamma}_{su(N,1)}$ can be written as $(n_p + n + \nu + 1)$, $(\nu + 1)$ and $N(N+1)[(\nu+1)/(N+1)][(\nu+1)/(N+1) - 1]$, respectively. In the same sense as that in the previous case, the present case is also related to the Holstein-Primakoff boson expansion in the symmetric representation of the $su(N,1)$-algebra, which is given as

$$\hat{T}^p = \hat{a}^{p*} \cdot \sqrt{\nu + 1 + \sum_{r=1}^{N} \hat{a}^{r*}\hat{a}^r},$$
$$\hat{T}_p = \sqrt{\nu + 1 + \sum_{r=1}^{N} \hat{a}^{r*}\hat{a}^r} \cdot \hat{a}^p,$$
$$\hat{T}_q^p = \delta^{pq} \left(\nu + 1 + \sum_{r=1}^{N} \hat{a}^{r*}\hat{a}^r \right) + \hat{a}^{p*}\hat{a}^q. \tag{4·6}$$

The case $N = 1$ corresponds to the $su(1,1)$-algebra.

4.3. *The case* $(M = 1,\ N = 1)$: *the* $su(2)$- *and* $su(1,1)$-*algebras*

Let us investigate the case $(M = 1,\ N = 1)$, i.e., $i = p = 1$, which can be constructed in terms of the boson operators $(\hat{a}_1, \hat{a}_1^*)$, $(\hat{a}^1 = \hat{a}, \hat{a}^{1*} = \hat{a}^*)$, $(\hat{b}_1^1 = \hat{b}_1, \hat{b}_1^{1*} = \hat{b}_1^*)$ and $(\hat{b}, \hat{b}^*)$. Then, the generators of the two algebras can be expressed by

$$\begin{aligned}
\hat{S}^1 &= \hat{a}_1^*\hat{b} + \hat{a}^*\hat{b}_1, \quad \hat{S}_1 = \hat{b}^*\hat{a}_1 + \hat{b}_1^*\hat{a}, \\
\hat{S}_1^1 &= (\hat{a}_1^*\hat{a}_1 - \hat{b}^*\hat{b}) + (\hat{a}^*\hat{a} - \hat{b}_1^*\hat{b}_1), && (4{\cdot}7\text{a}) \\
\hat{S} &= (\hat{a}_1^*\hat{a}_1 + \hat{b}^*\hat{b}) - (\hat{a}^*\hat{a} + \hat{b}_1^*\hat{b}_1), && (4{\cdot}7\text{b}) \\
\hat{T}^1 &= \hat{b}^*\hat{a}^* - \hat{b}_1^*\hat{a}_1^*, \quad \hat{T}_1 = \hat{a}\hat{b} - \hat{a}_1\hat{b}_1, \\
\hat{T}_1^1 &= (\hat{b}\hat{b}^* + \hat{a}^*\hat{a}) + (\hat{b}_1\hat{b}_1^* + \hat{a}_1^*\hat{a}_1), && (4{\cdot}8\text{a}) \\
\hat{T} &= (\hat{b}\hat{b}^* - \hat{a}^*\hat{a}) - (\hat{b}_1\hat{b}_1^* - \hat{a}_1^*\hat{a}_1). && (4{\cdot}8\text{b})
\end{aligned}$$

We can see that the $su(2)$- and the $su(1,1)$-generators given in the relations (4·7) and (4·8) consist of two sets of the $su(2)$- and the $su(1,1)$-spins, respectively. In the same sense as in the cases $(M = 1,\ N = 0)$ and $(M = 0,\ N = 1)$, the present form differs from the conventional form by a factor of 2. The operator $\hat{D}^{ip}$ can be expressed in the following form:

$$\hat{D}^{11} = \hat{b}_1^*\hat{b} + \hat{a}^*\hat{a}_1. \tag{4·9}$$

Hereafter, we use also the notation $\hat{R}^1$ for $\hat{D}^{11}$:

$$\hat{R}^1 = \hat{D}^{11}. \tag{4·9a}$$

Here, $\boldsymbol{n} = (n_1 = n)$, $\boldsymbol{m} = (m_1 = m)$ and $\boldsymbol{l} = (l_{11} = l)$, then, $\boldsymbol{k} = (l)$ and $C(\boldsymbol{k}, \boldsymbol{l}) = 1$. The state $|n, m, k; \nu\rangle$ is expressed as

$$\begin{aligned}
|n, m, l; \nu\rangle &= \hat{F}(n)\hat{E}(m)\hat{D}(l)|\nu\rangle \\
&= (\hat{F}^1)^n(\hat{E}^1)^m(\hat{D}^{11})^l|\nu\rangle \\
&= (\hat{T}^1)^n(\hat{S}^1)^m(\hat{R}^1)^l \cdot (\hat{b}^*)^\nu|0\rangle. && (4{\cdot}10)
\end{aligned}$$

In this case, also, it is not necessary to solve the eigenvalue equations (3·21c). The eigenvalues of $\hat{E}_1^1$, $\hat{E}$, $\hat{F}_1^1$, $\hat{F}$, $\hat{\Gamma}_{u(2)}$ and $\hat{\Gamma}_{u(1,1)}$ for the state $|n, m, l; \nu\rangle$ are given as $(m - l)$, $(m + l - \nu)$, $(n + l + 1)$, $(n + \nu - l + 1)$, $[l(l + 1) + (\nu - l)(\nu - l + 1)]$ and $[l(l+1) + (\nu - l)(\nu - l + 1)]$, respectively. Therefore, the eigenvalues of the operators $\hat{S}_1^1$, $\hat{S}$, $\hat{T}_1^1$, $\hat{T}$, $\hat{\Gamma}_{su(2)}$ and $\hat{\Gamma}_{su(1,1)}$ are expressed in the following forms:

$$\begin{aligned}
& s_1^1 = 2m - \nu = 2(m - \nu/2), \quad s = \nu - 2l = 2(\nu/2 - l), && (4{\cdot}11\text{a}) \\
& t_1^1 = 2n + \nu + 2 = 2(n + (\nu + 2)/2), \\
& t = \nu - 2l = 2(\nu/2 - l), && (4{\cdot}11\text{b}) \\
& \gamma_{su(2)} = 2(\nu/2)(\nu/2 + 1), \\
& \gamma_{su(1,1)} = 2[(\nu + 2)/2][(\nu + 2)/2 - 1]. && (4{\cdot}11\text{c})
\end{aligned}$$

We see that the results for s_1^1, t_1^1, $\gamma_{su(2)}$ and $\gamma_{su(1,1)}$ have well-known forms. The factor 2 comes from the definitions of $\hat{S}_1^1$ and $\hat{T}_1^1$, which differ from the conventional definitions by a factor of 2. This may be clear from the relations (2·12) and (2·13). In the present case, the number of mutually commuting operators $\{\hat{S}_1^1, \hat{T}_1^1, \hat{\Gamma}_{su(2)}, \hat{\Gamma}_{su(1,1)}\}$ is equal to that of the quantum numbers specifying the orthogonal basis. Therefore, the linearly independent basis automatically becomes orthogonal.

Our present problem is essentially the classification of a system constructed with four kinds of bosons. From the above argument, the meaning of the quantum numbers n, m and ν may be clear. As a final argument, let us make clear the meaning of the quantum number l. In association with $\hat{R}^1$ defined in Eq. (4·9) and $\hat{S}$, we introduce $\hat{R}_1$, which is hermite conjugate to $\hat{R}^1$, denote $\hat{S}$ as $-\hat{R}_1^1$, and introduce the new operator $\hat{R}$ defined as

$$\begin{aligned}
\hat{R}^1 &= \hat{b}_1^*\hat{b} + \hat{a}^*\hat{a}_1, \quad \hat{R}_1 = \hat{b}^*\hat{b}_1 + \hat{a}_1^*\hat{a}(= \hat{R}^{1*}), \\
\hat{R}_1^1 &= (\hat{b}_1^*\hat{b}_1 - \hat{b}^*\hat{b}) + (\hat{a}^*\hat{a} - \hat{a}_1^*\hat{a}_1)(= -\hat{S} = -\hat{E}_1^1 + \hat{E}), \qquad (4\cdot12\text{a}) \\
\hat{R} &= (\hat{b}_1^*\hat{b}_1 + \hat{b}^*\hat{b}) - (\hat{a}^*\hat{a} + \hat{a}_1^*\hat{a}_1)(= -\hat{E}_1^1 - \hat{E}). \qquad (4\cdot12\text{b})
\end{aligned}$$

We can prove that the set $\{\hat{R}^1, \hat{R}_1, \hat{R}_1^1\}$ forms an $su(2)$-algebra and that it commutes with $\hat{R}$. Furthermore, it commutes with the sets $\{\hat{S}^1, \hat{S}_1, \hat{S}_1^1\}$ and $\{\hat{T}^1, \hat{T}_1, \hat{T}_1^1\}$. The eigenvalues of $\hat{R}_1^1$, $\hat{R}$ and the Casimir operator for $\{\hat{R}^1, \hat{R}_1, \hat{R}_1^1\}$ with respect to the state (4·10), which we denote r_1^1, r and $\gamma'_{su(2)}$, are given as

$$\begin{aligned}
&r_1^1 = 2l - \nu = 2(l - \nu/2), \quad r = \nu - 2m = 2(\nu/2 - m), \qquad (4\cdot13\text{a}) \\
&\gamma'_{su(2)} = 2(\nu/2)(\nu/2 + 1). \qquad (4\cdot13\text{b})
\end{aligned}$$

Then, the meaning of the quantum number l is clear. For classification of a system composed of four kinds of bosons, the $su(2) \otimes su(2)$-scheme is well known. The present scheme is composed of two $su(2)$-spins and one $su(1,1)$-spin, and we can construct any state by successive operation of the raising operators $\hat{T}^1$, $\hat{S}^1$ and $\hat{R}^1$, which all commute, on the state $(\hat{b}^*)^\nu|0\rangle$. In a forthcoming paper, we will report on this point in more detail.

§5. The $su(3)$- and the $su(2,1)$-algebra

In this section, we investigate the cases ($M = 2$, $N = 1$) and ($M = 1$, $N = 2$). These two cases correspond to the $su(3)$- and the $su(2,1)$-algebra, and this section is a central part of this paper. Since $MN \leq M + N$, it is not necessary to contact with the eigenvalue problem of the Casimir operators.

5.1. *The case* ($M = 2$, $N = 1$): *the* $su(3)$- *and the* $su(1,1)$-*algebra*

This case is obtained by setting $i = 1, 2$ and $p = 1$ and given with the boson operators $(\hat{a}_1, \hat{a}_1^*)$, $(\hat{a}_2, \hat{a}_2^*)$, $(\hat{a}^1 = \hat{a}, \hat{a}^{1*} = \hat{a}^*)$, $(\hat{b}_1^1 = \hat{b}_1, \hat{b}_1^{1*} = \hat{b}_1^*)$, $(\hat{b}_2^1 = \hat{b}_2, \hat{b}_2^1{*} = \hat{b}_2^*)$ and $(\hat{b}, \hat{b}^*)$. Totally, we treat the case of six kinds of bosons. The generators of the two algebras are exressed in terms of the above bosons as

$$\hat{S}^1 = \hat{a}_1^*\hat{b} + \hat{a}^*\hat{b}_1, \quad \hat{S}^2 = \hat{a}_2^*\hat{b} + \hat{a}^*\hat{b}_2,$$

$$
\begin{aligned}
\hat{S}_1 &= \hat{b}^*\hat{a}_1 + \hat{b}_1^*\hat{a}, \quad \hat{S}_2 = \hat{b}^*\hat{a}_2 + \hat{b}_2^*\hat{a}, \\
\hat{S}_2^1 &= \hat{a}_2^*\hat{a}_1 - \hat{b}_1^*\hat{b}_2, \quad \hat{S}_1^2 = \hat{a}_1^*\hat{a}_2 - \hat{b}_2^*\hat{b}_1, \\
\hat{S}_1^1 &= (\hat{a}_1^*\hat{a}_1 - \hat{b}^*\hat{b}) + (\hat{a}^*\hat{a} - \hat{b}_1^*\hat{b}_1), \\
\hat{S}_2^2 &= (\hat{a}_2^*\hat{a}_2 - \hat{b}^*\hat{b}) + (\hat{a}^*\hat{a} - \hat{b}_2^*\hat{b}_2), && (5\cdot1\text{a}) \\
\hat{S} &= (\hat{a}_1^*\hat{a}_1 + \hat{a}_2^*\hat{a}_2 + \hat{b}^*\hat{b}) - (\hat{a}^*\hat{a} + \hat{b}_1^*\hat{b}_1 + \hat{b}_2^*\hat{b}_2). && (5\cdot1\text{b}) \\
\hat{T}^1 &= \hat{b}^*\hat{a}^* - \hat{b}_1^*\hat{a}_1^* - \hat{b}_2^*\hat{a}_2^*, \\
\hat{T}_1 &= \hat{a}\hat{b} - \hat{a}_1\hat{b}_1 - \hat{a}_2\hat{b}_2, \\
\hat{T}_1^1 &= (\hat{b}\hat{b}^* + \hat{a}^*\hat{a}) + (\hat{b}_1\hat{b}_1^* + \hat{a}_1^*\hat{a}_1) + (\hat{b}_2\hat{b}_2^* + \hat{a}_2^*\hat{a}_2), && (5\cdot2\text{a}) \\
\hat{T} &= (\hat{b}\hat{b}^* - \hat{a}^*\hat{a}) - (\hat{b}_1\hat{b}_1^* - \hat{a}_1^*\hat{a}_1) - (\hat{b}_2\hat{b}_2^* - \hat{a}_2^*\hat{a}_2). && (5\cdot2\text{b})
\end{aligned}
$$

We can see that the set of the $su(1,1)$-generators given in relation (5·2) consists of three sets of $su(1,1)$-spins. Further, the operators $\hat{D}^{ip}$ given in Eq. (3·9) can be expressed in the forms

$$\hat{D}^{11} = \hat{b}_1^*\hat{b} + \hat{a}^*\hat{a}_1, \quad \hat{D}^{21} = \hat{b}_2^*\hat{b} + \hat{a}^*\hat{a}_2. \tag{5·3}$$

Hereafter, we also use the notation $\hat{R}^1$ and $\hat{R}^2$ for $\hat{D}^{11}$ and $\hat{D}^{21}$:

$$\hat{R}^1 = \hat{D}^{11}, \quad \hat{R}^2 = \hat{D}^{21}. \tag{5·3a}$$

In this case, $\boldsymbol{n} = (n_1 = n)$, $\boldsymbol{m} = (m_1, m_2)$ and $\boldsymbol{l} = (l_{11} = l_1, l_{21} = l_2)$. Then, $\boldsymbol{k} = (l_{11} = l_1, l_{21} = l_2)$ and $C(\boldsymbol{k}, \boldsymbol{l}) = 1$. The state $|\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{k}; \nu\rangle$ is expressed as

$$
\begin{aligned}
|n, m_1, m_2, l_1, l_2; \nu\rangle &= \hat{F}(n)\hat{E}(m_1, m_2)\hat{D}(l_1, l_2)|\nu\rangle \\
&= (\hat{F}^1)^n(\hat{E}^1)^{m_1}(\hat{E}^2)^{m_2}(D^{11})^{l_1}(\hat{D}^{21})^{l_2}|\nu\rangle \\
&= (\hat{T}^1)^n(\hat{S}^1)^{m_1}(\hat{S}^2)^{m_2}(\hat{R}^1)^{l_1}(\hat{R}^2)^{l_2} \cdot (\hat{b}^*)^\nu|0\rangle. \qquad (5\cdot4)
\end{aligned}
$$

In this case, it is also not necessary to solve the eigenvalue equations (3·21c).

The eigenvalues of the operators $\hat{E}_1^1$, $\hat{E}_2^2$, $\hat{E}$, $\hat{F}_1^1$, $\hat{F}$, $\hat{\Gamma}_{u(3)}$ and $\hat{\Gamma}_{u(1,1)}$ are given as $(m_1 - l_1)$, $(m_2 - l_2)$, $(m_1 + m_2 + l_1 + l_2 - \nu)$, $(n + l_1 + l_2 + 2)$, $(n + \nu - l_1 - l_2 + 1)$, $2(l_1 + l_2)^2 - 2\nu(l_1 + l_2) + \nu(\nu + 2)$ and $2(l_1 + l_2)^2 - 2(\nu - 1)(l_1 + l_2) + (\nu^2 + \nu + 2)$. Therefore, the eigenvalues of the operators $\hat{S}_1^1$, $\hat{S}_2^2$, $\hat{S}$, $\hat{T}_1^1$, $\hat{T}$, $\hat{\Gamma}_{su(3)}$ and $\hat{\Gamma}_{su(1,1)}$ are obtained in the following forms:

$$
\begin{aligned}
& s_1^1 = 2m_1 + m_2 + l_2 - \nu, \quad s_2^2 = m_1 + 2m_2 + l_1 - \nu, \\
& s = \nu - 2(l_1 + l_2), && (5\cdot5\text{a}) \\
& t_1^1 = 2(n + (\nu + 3)/2), \quad t = \nu - 2(l_1 + l_2) - 1, && (5\cdot5\text{b})
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{su(3)} &= (2/3)[\nu - (l_1 + l_2)/2][\nu - (l_1 + l_2)/2 + 3] \\
&\quad + 2[(l_1 + l_2)/2][(l_1 + l_2)/2 + 1], \\
\gamma_{su(1,1)} &= 2[(\nu + 3)/2][(\nu + 3)/2 - 1]. && (5\cdot6\text{a})
\end{aligned}
$$

As can be seen in the relation (5·6a), λ and μ appearing in Elliott's model correspond to the forms $\lambda = \nu - (l_1 + l_2)/2$ and $\mu = (l_1 + l_2)/2$, respectively.[1)]

As mentioned in §3, the states (5·4) do not always form an orthogonal set. In order to get the orthogonal set from the states (5·4), we introduce the $su(2)$-spin in the following form:

$$\hat{\sigma}_+ = \hat{S}_2^1, \quad \hat{\sigma}_- = \hat{S}_1^2, \quad \hat{\sigma}_0 = (\hat{S}_2^2 - \hat{S}_1^1)/2, \tag{5·7}$$

$$\hat{\boldsymbol{\sigma}}^2 = (\hat{\sigma}_+\hat{\sigma}_- + \hat{\sigma}_-\hat{\sigma}_+)/2 + \hat{\sigma}_0^2. \tag{5·7a}$$

Further, the operator $\hat{\tau}$ is introduced in the form

$$\hat{\tau} = (\hat{S}_2^2 + \hat{S}_1^1)/2. \tag{5·8}$$

Then, for the present system, we are able to prepare six hermitian operators, $\hat{\Gamma}_{su(1,1)}$, $\hat{T}_1^1$, $\hat{\Gamma}_{su(3)}$, $\hat{\tau}$, $\hat{\boldsymbol{\sigma}}^2$ and $\hat{\sigma}_0$, which all commute. First, it should be noted that the operators $(\hat{S}^1, \hat{S}^2)$ and $(\hat{R}^2, -\hat{R}^1)$ are regarded as spherical tensors of rank $j = 1/2$ for the $su(2)$-spin $\{\hat{\sigma}_{\pm,0}\}$, i.e., $(\hat{S}^1 = \hat{S}_{1/2,-1/2}, \hat{S}^2 = \hat{S}_{1/2,+1/2})$ and $(\hat{R}^2 = \hat{R}_{1/2,-1/2}, -\hat{R}^1 = \hat{R}_{1/2,+1/2})$. If changing the notation such that $(m_1 = m - m_0,\ m_2 = m + m_0)$ and $(l_1 = l + l_0,\ l_2 = l - l_0)$, we have the forms $(\hat{S}^1)^{m_1} \cdot (\hat{S}^2)^{m_2} = (\hat{S}_{1/2,-1/2})^{m-m_0} \cdot (\hat{S}_{1/2,+1/2})^{m+m_0}$ and $(\hat{R}^1)^{l_1} \cdot (\hat{R}^2)^{l_2} = (-)^{l+l_0} \cdot (\hat{R}_{1/2,+1/2})^{l+l_0} \cdot (\hat{R}_{1/2,-1/2})^{l-l_0}$. The state $(\hat{b}^*)^\nu|0\rangle$ is a zero spin state. Then, we can introduce the state $|n, m, l, k, k_0; \nu\rangle\rangle$, which is connected with the state (5·4) in the following form:

$$|n, m, l, k, k_0; \nu\rangle\rangle = \sum_{m_0 l_0} \langle m, m_0, l, l_0|k, k_0\rangle(-)^{l+l_0} \cdot |n, m - m_0, m + m_0, l + l_0, l - l_0; \nu\rangle. \tag{5·9}$$

Here, $\langle m, m_0, l, l_0|k, k_0\rangle$ denotes the Clebsch-Gordan coefficient. The state $|n, m, l, k, k_0; \nu\rangle\rangle$ is an eigenstate of the six hermitian operators $\hat{\Gamma}_{su(1,1)}$, $\hat{T}_1^1$, $\hat{\Gamma}_{su(3)}$, $\hat{\tau}$, $\hat{\boldsymbol{\sigma}}^2$ and $\hat{\sigma}_0$ with the eigenvalues $2[(\nu+3)/2][(\nu+3)/2-1]$, $2[n+(\nu+3)/2]$, $[(2/3)\cdot(\nu-l)(\nu-l+3)+2l(l+1)]$, $(3m+l-\nu)$, $k(k+1)$ and k_0, respectively. Therefore, we are able to conclude that $\{|n, m, l, k, k_0; \nu\rangle\rangle\}$ is an orthogonal set. Of course, the normalized state $|n, m - m_0, m + m_0, l + l_0, l - l_0; \nu\rangle$ should be used. As a remark, we point out that we can adopt the operator $\hat{S}$ given in Eq. (5·1b) instead of the Casimir operator $\hat{\Gamma}_{su(3)}$ for mutually commuting operators. In this case, the eigenvalue is $(\nu - 4l)$.

As a final investigation of this case, we consider the relation with the operators $\hat{R}^1$ and $\hat{R}^2$ defined in Eq. (5·3a). In association with the operators $\hat{R}^1$, $\hat{R}^2$ and $\hat{S}$ given in Eq. (5·1b), we introduce $\hat{R}_1$ and $\hat{R}_2$, which are complex conjugates of $\hat{R}^1$ and $\hat{R}^2$, and $\hat{R}_2^1$, $\hat{R}_1^2$, $\hat{R}_1^1$, $\hat{R}_2^2$ and $\hat{R}$:

$$\begin{aligned}
\hat{R}^1 &= \hat{b}_1^*\hat{b} + \hat{a}^*\hat{a}_1, \quad \hat{R}^2 = \hat{b}_2^*\hat{b} + \hat{a}^*\hat{a}_2, \\
\hat{R}_1 &= \hat{b}^*\hat{b}_1 + \hat{a}_1^*\hat{a}\ (= \hat{R}^{1*}), \quad \hat{R}_2 = \hat{b}^*\hat{b}_2 + \hat{a}_2^*\hat{a}\ (= \hat{R}^{2*}), \\
\hat{R}_2^1 &= \hat{b}_2^*b_1 - \hat{a}_1^*\hat{a}_2 = -\hat{S}_1^2, \quad \hat{R}_1^2 = \hat{b}_1^*\hat{b}_2 - \hat{a}_2^*\hat{a}_1 = -\hat{S}_2^1, \\
\hat{R}_1^1 &= (\hat{b}_1^*\hat{b}_1 - \hat{b}^*\hat{b}) + (\hat{a}^*\hat{a} - \hat{a}_1^*\hat{a}_1) \quad (= \hat{E}_2^2 - \hat{S} = -\hat{E}_1^1 + \hat{E}), \\
\hat{R}_2^2 &= (\hat{b}_2^*\hat{b}_2 - \hat{b}^*\hat{b}) + (\hat{a}^*\hat{a} - \hat{a}_2^*\hat{a}_2) \\
(&= \hat{E}_1^1 - \hat{S} = -\hat{E}_2^2 + \hat{E}),
\end{aligned} \tag{5·10a}$$

$$\begin{aligned}
\hat{R} &= (\hat{b}_1^*\hat{b}_1 + \hat{b}_2^*\hat{b}_2 + \hat{b}^*\hat{b}) - (\hat{a}^*\hat{a} + \hat{a}_1^*\hat{a}_1 + \hat{a}_2^*\hat{a}_2) \\
(&= -\hat{E}_1^1 - \hat{E}_2^2 - \hat{E}).
\end{aligned} \tag{5·10b}$$

We can prove that the set $\{\hat{R}^1, \hat{R}^2, \hat{R}_1, \hat{R}_2, \hat{R}_2^1, \hat{R}_1^2, \hat{R}_1^1, \hat{R}_2^2\}$ forms an $su(3)$-algebra and that it commutes with $\hat{R}$. Further, it commutes with $\{\hat{T}^1, \hat{T}_1, \hat{T}_1^1\}$, but it does not commute with $\hat{S}^1$, etc. The state (5·9) is also an eigenstate of the operators $\hat{R}_1^1$, $\hat{R}_2^2$, $\hat{R}$ and $\hat{\Gamma}'_{su(3)}$, and the eigenvalues, which we denote by r_1^1, r_2^2, r and $\gamma'_{su(3)}$, are given by

$$r_1^1 = 2l_1 + l_2 + m_2 - \nu, \quad r_2^2 = l_1 + 2l_2 + m_1 - \nu, \quad r = \nu - 2(m_1 + m_2), \quad (5\cdot5c)$$

$$\gamma'_{su(3)} = (2/3)[\nu-(m_1+m_2)/2][\nu-(m_1+m_2)/2+3]+2[(m_1+m_2)/2][(m_1+m_2)/2+1]. \quad (5\cdot6b)$$

From the above argument, we find that there exist two Schwinger boson representations for the $su(3)$-algbra in the system with six kinds of boson operators. The eigenstate is constructed by successive operations of the operators $\hat{T}^1$, $\hat{S}^1$ and $\hat{S}^2$ on the state $(\hat{R}^1)^{l_1}(\hat{R}^2)^{l_2} \cdot (\hat{b}^*)^\nu 0\rangle$, and also successive operations of the operators $\hat{T}^1$, $\hat{R}_1$ and $\hat{R}^2$ on the state $(\hat{S}^1)^{m_1}(\hat{S}^2)^{m_2}(\hat{b}^*)^\nu|0\rangle$. It should be noted that $\{\hat{T}^1, \hat{T}_1, \hat{T}_1^1\}$ commutes with all generators of the two $su(3)$-algebras.

5.2. *The case* ($M = 1$, $N = 2$): *the* $su(2)$- *and the* $su(2,1)$-*algebra*

This case is obtained by setting $i = 1$ and $p = 1, 2$ and given with the bosons $(\hat{a}_1 = \hat{a}, \hat{a}_1^* = \hat{a}^*)$, $(\hat{a}^1, \hat{a}^{1*})$, $(\hat{a}^2, \hat{a}^{2*})$, $(\hat{b}_1^1 = \hat{b}_1, \hat{b}_1^{1*} = \hat{b}^{1*})$, $(\hat{b}_1^2 = \hat{b}^2, \hat{b}_1^{2*} = \hat{b}^{2*})$ and $(\hat{b}, \hat{b}^*)$. The generators of the two algebras can be written down in the forms shown as follows:

$$\begin{aligned}
\hat{S}^1 &= \hat{a}^*\hat{b} + \hat{a}^{1*}\hat{b}^1 + \hat{a}^{2*}\hat{b}^2, \quad \hat{S}_1 = \hat{b}^*\hat{a} + \hat{b}^{1*}\hat{a}^1 + \hat{b}^{2*}\hat{a}^2, \\
\hat{S}_1^1 &= (\hat{a}^*\hat{a} - \hat{b}^*\hat{b}) + (\hat{a}^{1*}\hat{a}^1 - \hat{b}^{1*}\hat{b}^1) + (\hat{a}^{2*}\hat{a}^2 - \hat{b}^{2*}\hat{b}^2), && (5\cdot11a) \\
\hat{S} &= (\hat{a}^*\hat{a} + \hat{b}^*\hat{b}) - (\hat{a}^{1*}\hat{a}^1 + \hat{b}^{1*}\hat{b}^1) - (\hat{a}^{2*}\hat{a}^2 + \hat{b}^{2*}\hat{b}^2), && (5\cdot11b) \\
\hat{T}^1 &= \hat{b}^*\hat{a}^{1*} - \hat{b}^{1*}\hat{a}^*, \quad \hat{T}^2 = \hat{b}^*\hat{a}^{2*} - \hat{b}^{2*}\hat{a}^*, \\
\hat{T}_1 &= \hat{a}^1\hat{b} - \hat{a}\hat{b}^1, \quad \hat{T}_2 = \hat{a}^2\hat{b} - \hat{a}\hat{b}^2, \\
\hat{T}_2^1 &= \hat{b}^2\hat{b}^{1*} + \hat{a}^{1*}\hat{a}^2, \quad \hat{T}_1^2 = \hat{b}^1\hat{b}^{2*} + \hat{a}^{2*}\hat{a}^1, \\
\hat{T}_1^1 &= (\hat{b}\hat{b}^* + \hat{a}^{1*}\hat{a}^1) + (\hat{b}^1\hat{b}^{1*} + \hat{a}^*\hat{a}), \\
\hat{T}_2^2 &= (\hat{b}\hat{b}^* + \hat{a}^{2*}\hat{a}^2) + (\hat{b}^2\hat{b}^{2*} + \hat{a}^*\hat{a}), && (5\cdot12a) \\
\hat{T} &= (\hat{b}\hat{b}^* - \hat{a}^{1*}\hat{a}^1 - \hat{a}^{2*}\hat{a}^2) - (\hat{b}^1\hat{b}^{1*} + \hat{b}^2\hat{b}^{2*} - \hat{a}^*\hat{a}). && (5\cdot12b)
\end{aligned}$$

The set of $su(2)$-generators (5·11) consists of three sets of the $su(2)$-spins. Further, the $\hat{D}^{ip}$ in Eq. (3·9) are expressed as

$$\hat{D}^{11} = \hat{b}^{1*}\hat{b} + \hat{a}^{1*}\hat{a}, \quad \hat{D}^{12} = \hat{b}^{2*}\hat{b} + \hat{a}^{2*}\hat{a}. \quad (5\cdot13)$$

Hereafter, we also use the notation $\hat{R}^1$ and $\hat{R}^2$ for $\hat{D}^{11}$ and $\hat{D}^{12}$:

$$\hat{R}^1 = \hat{D}^{11}, \quad \hat{R}^2 = \hat{D}^{12}. \quad (5\cdot13a)$$

In this case, $\boldsymbol{n} = (n_1, n_2)$, $\boldsymbol{m} = (m_1 = m)$ and $\boldsymbol{l} = (l_{11} = l_1, l_{12} = l_2)$, and, therefore, $\boldsymbol{k} = (l_{11} = l_1, l_{12} = l_2)$ and $C(\boldsymbol{k}, \boldsymbol{l}) = 1$. The state $|n_1, n_2, m, l_1, l_2; \nu\rangle$ can be expressed as

$$|n_1, n_2, m, l_1, l_2; \nu\rangle = \hat{F}(n_1, n_2)\hat{E}(m)\hat{D}(l_1, l_2)|\nu\rangle$$

$$= (\hat{F}^1)^{n_1}(\hat{F}^2)^{n_2}(\hat{E}^1)^m(\hat{D}^{11})^{l_1}(\hat{D}^{12})^{l_2}|\nu\rangle$$
$$= (\hat{T}^1)^{n_1}(\hat{T}^2)^{n_2}(\hat{S}^1)^m(\hat{R}^1)^{l_1}(\hat{R}^2)^{l_2}(\hat{b}^*)^\nu|0\rangle. \qquad (5\cdot14)$$

In this case, also, it is not necessary to solve Eq. (3·21c).

The eigenvalues of $\hat{E}^1_1$, $\hat{E}$, $\hat{F}^1_1$, $\hat{F}^2_2$, $\hat{F}$, $\hat{\Gamma}_{su(2)}$ and $\hat{\Gamma}_{su(2,1)}$ are given as $(m-l_1-l_2)$, $(m+l_1+l_2-\nu)$, (n_1+1+l_1), (n_2+1+l_2), $(n_1+n_2+\nu+1-l_1-l_2)$, $2(l_1+l_2)^2-2\nu\cdot(l_1+l_2)+\nu(\nu+1)$ and $2(l_1+l_2)^2-2(\nu-1)(l_1+l_2)+\nu^2-1$. Then, the eigenvalues of $\hat{S}^1_1$, $\hat{S}$, $\hat{T}^1_1$, $\hat{T}^2_2$, $\hat{T}$, $\hat{\Gamma}_{su(2)}$ and $\hat{\Gamma}_{su(2,1)}$, are given by

$$s^1_1 = 2(m-\nu/2), \quad s = 2(\nu/2-(l_1+l_2)), \qquad (5\cdot15a)$$
$$t^1_1 = 2n_1+n_2-l_2+\nu+2, \quad t^2_2 = n_1+2n_2-l_1+\nu+2,$$
$$t = \nu-2(l_1+l_2)-1, \qquad (5\cdot15b)$$
$$\gamma_{su(2)} = 2(\nu/2)(\nu/2+1),$$
$$\gamma_{su(2,1)} = (2/3)[\nu-(l_1+l_2+2)/2][\nu-(l_1+l_2+2)/2+3] + 2[(l_1+l_2+2)/2][(l_1+l_2+2)/2-1]. \qquad (5\cdot16a)$$

The states (5·14) also do not always form an orthogonal set. In a manner similar to that in the case of the $su(3)$-algebra, the orthogonal set is obtained. The above-described process is analogous to that in the previous case.

As in the previous case, let us discuss the role of the operators $\hat{R}^1$ and $\hat{R}^2$. In addition to these two operators, we introduce $\hat{R}_1$, $\hat{R}_2$, $\hat{R}^1_2$, $\hat{R}^2_1$, $\hat{R}^1_1$, $\hat{R}^2_2$ and $\hat{R}$ in the following forms:

$$\begin{aligned}
\hat{R}^1 &= \hat{b}^{1*}\hat{b}+\hat{a}^{1*}\hat{a}, \quad \hat{R}^2 = \hat{b}^{2*}\hat{b}+\hat{a}^{2*}\hat{a},\\
\hat{R}_1 &= \hat{b}^*\hat{b}^1+\hat{a}^*\hat{a}^1 \ (=\hat{R}^{1*}), \quad \hat{R}_2 = \hat{b}^*\hat{b}^2+\hat{a}^*\hat{a}^2 \ (=\hat{R}^{2*}),\\
\hat{R}^1_2 &= \hat{b}^{2*}\hat{b}^1+\hat{a}^{2*}\hat{a}^1 \ (=\hat{T}^2_1), \quad \hat{R}^2_1 = \hat{b}^{1*}\hat{b}^2+\hat{a}^{1*}\hat{a}^2 \ (=\hat{T}^1_2),\\
\hat{R}^1_1 &= (\hat{b}^{1*}\hat{b}^1-\hat{b}^*\hat{b})+(\hat{a}^{1*}\hat{a}^1-\hat{a}^*\hat{a}) \quad (=-\hat{F}^2_2-\hat{T}=\hat{F}^1_1-\hat{F}),\\
\hat{R}^2_2 &= (\hat{b}^{2*}\hat{b}^2-\hat{b}^*\hat{b})+(\hat{a}^{2*}\hat{a}^2-\hat{a}^*\hat{a})\\
(&= -\hat{F}^1_1-\hat{T}=\hat{F}^2_2-\hat{F}), \qquad (5\cdot17a)\\
\hat{R} &= (\hat{b}^{1*}\hat{b}^1+\hat{b}^{2*}\hat{b}^2+\hat{b}^*\hat{b})+(\hat{a}^{1*}\hat{a}^1+\hat{a}^{2*}\hat{a}^2+\hat{a}^*\hat{a})\\
(&= \hat{F}^1_1+\hat{F}^2_2+\hat{F}-3). \qquad (5\cdot17b)
\end{aligned}$$

The set $\{\hat{R}^1,\hat{R}^2,\hat{R}_1,\hat{R}_2,\hat{R}^1_2,\hat{R}^2_1,\hat{R}^1_1,\hat{R}^2_2\}$ forms an $su(3)$-algebra and commutes with $\hat{R}$ and also with $\hat{S}^1$, $\hat{S}_1$ and $\hat{S}^1_1$, but it does not commute with $\hat{T}^1$, etc. The state (5·11) is also an eigenstate of the operators $\hat{R}^1_1$, $\hat{R}^2_2$, $\hat{R}$ and $\hat{\Gamma}'_{su(3)}$, and the eigenvalues which we denote r^1_1, r^2_2, r and $\gamma'_{su(3)}$ are expressed as

$$r^1_1 = 2l_1+l_2-n_2-\nu, \quad r^2_2 = l_1+2l_2-n_1-\nu, \quad r = \nu+2(n_1+n_2), \qquad (5\cdot15c)$$

$$\gamma'_{su(3)} = (2/3)[\nu-(n_1+n_2)/2][\nu-(n_1+n_2)/2+3]+2[(n_1+n_2)/2][(n_1+n_2)/+1]. \qquad (5\cdot16b)$$

From the above argument, we know that there exists one more Schwinger represenation of the $su(3)$-algebra. The eigenstate (5·11) is constructed by successive

operations of $\hat{T}^1$, $\hat{T}^2$ and $\hat{S}^1$ on the state $(\hat{R}^1)^{l_1}(\hat{R}^2)^{l_2}(\hat{b}^*)^\nu|0\rangle$ and successive operations of $\hat{S}^1$, $\hat{R}^1$ and $\hat{R}^2$ on $(\hat{T}^1)^{n_1}(\hat{T}^2)^{n_2}(\hat{b}^*)^\nu|0\rangle$. Here, it should be noted that $\{\hat{S}^1, \hat{S}_1, \hat{S}_1^1\}$ commutes with all generators of the $su(2,1)$- and the $su(3)$-algebra.

§6. Discussion and a concluding remark

In Ref. 14), we investigated three aspects of the $su(2,1)$-boson model: the mixed-mode coherent sate, the classical counterpart, and its requantization. Also, in Ref. 15), we applied this model to a possible description of a statistically mixed state in a form slightly modified from that given in Ref. 14). This $su(2,1)$-boson model consists of three kinds of bosons and, in the language presented in this paper, it corresponds to the case ($M = 0$, $N = 2$). Various relations are obtained by fixing the arbitrary integer N to 2 in the relations given in §4.2. In this case, the orthogonal basis is specified by three quantum number, and in Ref. 14), various forms are presented. One of these forms is used in Ref. 15), where interesting results are obtained. However, that form is different from the form given in this paper. On the other hand, the $su(3)$-algebra suggests that the orthogonal basis based on the $su(2,1)$-algebra may be specified by five quantum numbers in the case of a possible full representation. The case ($M = 1$, $N = 2$) discussed in §5.2 seems to support this point. This case gives us an orthogonal basis which is specified by six quantum numbers. However, successive operations of $\hat{S}^1$ do not give any change in the representation of the $su(2,1)$-algebra, because the set $\{\hat{S}^1, \hat{S}_1, \hat{S}_1^1\}$, expressed in terms of the six kinds of bosons, forms an $su(2)$-algebra and commutes with all generators of the $su(2,1)$-algebra. Therefore, concerning the $su(2,1)$-algebra, the states obtained by operation of $\hat{S}^1$ are equivalent. This implies that the orthogonal basis given in the $su(2,1)$-algebra is specified by five quantum numbers. A possible form of the orthogonal basis is obtained from the form given in relation (5·14). Translation of the basic idea for constructing the orthogonal basis adopted in Ref. 15) gives us the following idea: In Refs. 14) and 15), we used three kinds of bosons in the present notation, $(\hat{a}, \hat{a}^*)$, $(\hat{c}, \hat{c}^*)$ and $(\hat{d}, \hat{d}^*)$, which correspond to the present ones according to

$$(\hat{a}, \hat{a}^*) \to (\hat{a}, \hat{a}^*), \quad (\hat{c}, \hat{c}^*) \to (\hat{b}^1, \hat{b}^{1*}), \quad (\hat{d}, \hat{d}^*) \to (\hat{b}^2, \hat{b}^{2*}). \tag{6·1}$$

In addition to the above boson operators, $(\hat{b}, \hat{b}^*)$, $(\hat{a}^1, \hat{a}^{1*})$ and $(\hat{a}^2, \hat{a}^{2*})$ are introduced in the present $su(2,1)$-algebra. Therefore, the generators in the notation used in Refs. 14) and 15), $\hat{S}_{\pm,0}$, $\hat{T}_{\pm,0}$ and $\hat{R}_{\pm,0}$ are changed into the following:

$$\begin{aligned}
&\hat{S}_+ = \hat{T}_2^1, \quad \hat{S}_- = \hat{T}_1^2, \quad \hat{S}_0 = (\hat{T}_1^1 - \hat{T}_2^2)/2, \\
&\hat{T}_+ = -\hat{T}^1, \quad \hat{T}_- = -\hat{T}_1, \quad \hat{T}_0 = \hat{T}_1^1/2, \\
&\hat{R}_+ = -\hat{T}^2, \quad \hat{R}_- = -\hat{T}_2, \quad \hat{R}_0 = \hat{T}_2^2/2.
\end{aligned} \tag{6·2}$$

We can choose $\hat{T}^1$ and $\hat{T}_2^1$, which correspond to $\hat{T}_+$ and $\hat{S}_+$ in Refs. 14) and 15), respectively, as the raising operators, and the state $|\nu_b\rangle$ obeying $\hat{T}_1|\nu_b\rangle = \hat{T}_1^2|\nu_b\rangle = 0$ is of the form

$$|\nu_b\rangle = (\hat{T}^2)^{n_2}(\hat{S}^1)^m(\hat{R}_2)^l(\hat{b}^{2*})^\nu|0\rangle. \tag{6·3}$$

Therefore, we can construct the state

$$|n_1, n_{12}, n_2, m, l; \nu\rangle = (\hat{T}^1)^{n_1}(\hat{T}^1_2)^{n_{12}}(\hat{T}^2)^{n_2}(\hat{S}^1)^m(\hat{R}_2)^l(\hat{b}^{2*})^\nu|0\rangle. \qquad (6\cdot4)$$

It may be interesting to apply the above form to the statistically mixed state.

In this paper, we did not contact with the problem of solving the eigenvalue equation (3·21c) explicitly. We mentioned that it is not necessary to solve Eq. (3·21c) in the case $MN \leq M + N$. Concerning this condition, for example, let us consider the case $M = 3$, i.e., the $su(4)$-algebra. For the full representation of the $su(4)$-algebra, nine quantum numbers are necessary. In other words, at least nine kinds of bosons are necessary. In our present treatment for this case, the number of the kinds of bosons, $(M + 1)(N + 1)$, is equal to $4(N + 1)$, and we have the inequality $4(N + 1) \geq 9$, i.e., $N \geq 2$. Therefore, we should have $N = 2$ and $MN = 8$ and $M+N = 6$. Thus, in this case $MN > M+N$, and it is necessary to solve Eq. (3·21c). In a forthcoming paper, we will treat such cases.

The $su(4)$-algebra gives us interesting physics. Concerning a unification of superconductivity and antiferromagnetism in high T_c-superconductivity, the $so(5)$-model has been proposed.[16] Further, the $su(4)$-model was proposed for treating not only the $so(5)$-scheme but also the $so(4)$- and the $su(2) \otimes su(2)$-scheme.[17] In this sense, it may be interesting to investigate the $so(5)$-algebra, together with the $so(4)$- and the $su(2) \otimes su(2)$-algebra, which are subalgebras of the $su(4)$-algebra, in the prersent framework. This will be presented in the near future.

By combining with the mixed-mode coherent state, we obtain the classical counterpart, and then, by solving the variational equation for the time-dependent Schrödinger equation, we are able to describe the time-development of the mixed-mode coherent state.

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References

1) J. P. Elliott, Proc. R. Soc. London **A245** (1958), 128, 562.
2) T. Marumori, M. Yamamura, Y. Miyanishi and S. Nishiyama, Prog. Theor. Phys. Suppl. Extra Number (1968), 179.
C. Dasso, F. Krejs, A. Klein and D. K. Chattopadhyay, Nucl. Phys. **A210** (1971), 429.
D. Janssen, R. V. Jolos and F. Donau, Nucl. Phys. **A224** (1974), 93.
A. Arima and F. Iachello, Ann. of Phys. **99** (1976), 253.
3) A. Klein and E. R. Marshalek, Rev. Mod. Phys. **63** (1991), 375.
4) Special issue: International Conference on the Physics of Nuclear Structure at the Extremes, Lewes, Sussex, UK. 17-19, June 1998, J. of Phys. **G25** (1999), 577.
5) H. Ui, Prog. Theor. Phys. **44** (1970), 153.
6) D. J. Rowe, J. Math. Phys. **25** (1984), 2662.
7) E. Celeghini, M. Rasetti, M. Tarlini and G. Vitiello, Mod. Phys. Lett. **B3** (1989), 1213.
E. Celeghini, M. Rasetti and G. Vitiello, Ann. of Phys. **215** (1992), 156.

8) J. Schwinger, in *Quantum Theory of Angular Momentum*, ed. L. Biedenharn and H. Van Dam (Academic Press, New York, 1955), p. 229.
9) Y. Tsue, A. Kuriyama and M. Yamamura, Prog. Theor. Phys. **91** (1994), 49, 469.
10) T. Marumori, T. Maskawa, F. Sakata and A. Kuriyama, Prog. Theor. Phys. **64** (1980), 1294.
M. Yamamura and A. Kuriyama, Prog. Theor. Phys. Suppl. No. 93 (1987).
11) Y. Tsue, A. Kuriyama and M. Yamamura, Prog. Theor. Phys. **93** (1995), 541, 1037.
12) Y. Takahashi and H. Umezawa, Collect. Phenom. **2** (1975), 55.
H. Umezawa, H. Matsumoto and M. Tachiki, *Thermo Field Dynamics and Condensed States* (North Holland, Amsterdam, 1982).
13) A. Kuriyama, J. da Providência, Y. Tsue and M. Yamamura, Prog. Theor. Phys. **96** (1996), 387; **98** (1997), 381, 893.
14) A. Kuriyama, J. da Providência, Y. Tsue and M. Yamamura, Prog. Theor. Phys. **100** (1998), 993.
15) A. Kuriyama, J. da Providência, Y. Tsue and M. Yamamura, Prog. Theor. Phys. **100** (1998), 1181.
16) S.-C. Zhang, Science **275** (1997), 1089.
17) M. W. Guidry, L.-A. Wu, Y. Sun and C.-L. Wu, condmat/9903150; condmat/9905436; condmat/9912072.