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Quantum Equivalence of Auxiliary Field Methods in Supersymmetric Theories

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Quantum corrections to Legendre transformations are shown to cancel to all orders in supersymmetric theories in path integral formalism. Using this result, Lagrangians for auxiliary fields are generalized to non-quadratic forms. In supersymmetric effective nonlinear Lagrangians, the arbitrariness due to the existence of quasi Nambu-Goldstone bosons is shown to disappear when local auxiliary gauge fields are introduced.

§1. Introduction

The Legendre transformation plays an important role both in classical and quantum physics. It appears when we change independent variables in classical mechanics. In quantum physics, it is associated with the Fourier transformation. For example, let us consider the path integral

$$I = \int [d\sigma] \exp\left[i \int d^4x \, (\sigma^i \Phi_i - W(\sigma^1, \cdots, \sigma^n)) \right], \tag{1·1}$$

where $\sigma^i(x)$ and $\Phi_i(x)$ $(i = 1, \cdots, n)$ are real scalar fields and $W(\sigma^1, \cdots, \sigma^n)$ is an arbitrary function. In order to evaluate this integral, we expand the integrand around the stationary path $\hat{\sigma}$ defined by

$$\frac{\partial}{\partial \sigma^i}(\sigma^j \Phi_j - W(\sigma^1 \cdots \sigma^n))|_{\sigma^i = \hat{\sigma}^i} = \Phi_i - W_i(\hat{\sigma}^1, \cdots, \hat{\sigma}^n) = 0, \tag{1·2}$$

where the subscript i denotes differentiation with respect to σ^i. Then the integral can be performed by the saddle point method:

$$\begin{aligned} I &= \exp\left[i \int d^4x \left(\hat{\sigma}^i \Phi_i - W(\hat{\sigma}^1, \cdots, \hat{\sigma}^n) \right) \right] \\ &\times \int [d\sigma] \exp\left[-i \int d^4x \sum_{ij} \frac{1}{2} \frac{\partial^2 W(\sigma^1, \cdots, \sigma^n)}{\partial \sigma^i \partial \sigma^j} \bigg|_{\sigma^i = \hat{\sigma}^i} (\sigma - \hat{\sigma})^i (\sigma - \hat{\sigma})^j + \cdots \right]. \end{aligned} \tag{1·3}$$

Here the first factor is the classical contribution and the second factor consists of quantum corrections. Linear terms are absent because of Eq. (1·2). The quadratic terms provide a factor of $(\det \boldsymbol{W}(\hat{\sigma}))^{-\frac{1}{2}}$ after the Gaussian integration, with $(\boldsymbol{W})_{ij} =$

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$\frac{\partial^2 W}{\partial\sigma^i\partial\sigma^j}$ being the Hesse matrix. The remaining terms contribute to higher order quantum corrections. Thus, if we define $U(\Phi_i)$ by

$$I = \exp\left[iU(\Phi_i)\right], \tag{1·4}$$

then the classical expression

$$U(\Phi_i) = \hat{\sigma}^j\Phi_j - W(\hat{\sigma}^1, \cdots, \hat{\sigma}^n) \tag{1·5}$$

is just the lowest order approximation. Unless $W(\sigma)$ is a quadratic form, there are many higher order quantum corrections in general. In supersymmetric theories, however, we show this *classical expression is exact to all orders* because of remarkable cancellations of quantum corrections due to the supersymmetry.

This quantum Legendre transform may have many applications. For example, it can be applied to studies of supersymmetric nonlinear sigma models.[1)] They can be defined from linear sigma models by introducing auxiliary fields as Lagrange multipliers. For example, consider a Lagrangian $\mathcal{L} = \int d^4\theta K_0(\phi, \phi^\dagger) + (\int d^2\theta P(\phi) +$ c.c.), with the canonical Kähler potential $K_0 = \phi^\dagger\phi$ and the superpotential $P = \phi_0 g(\phi)$. Here, ϕ_0 is an auxiliary field without kinetic terms. The path integral over ϕ_0 gives an F-term constraint, $g(\phi) = 0$. By solving this equation and substituting the solution into K_0, we obtain the Kähler potential for a nonlinear sigma model. If we consider the above "linear model" as an effective theory, we have to use the more general Kähler potential $K_0 = f(\phi^\dagger\phi)$, where f is an arbitrary function.[2)] Then the resulting Kähler potential is $K = f(\phi^\dagger\phi)|_{g(\phi)=0}$ [3),4)] after integrating out ϕ_0.

In order to obtain a supersymmetric nonlinear sigma model on a *compact* manifold, we have to consider a gauged linear sigma model by introducing a vector superfield V as an auxiliary field:[*)] $K_0 = e^V\phi^\dagger\phi - cV$, where c is called a Fayet-Iliopoulos parameter. By solving the classical equation of motion for V, $e^V\phi^\dagger\phi - c = 0$, and substituting back into K_0, we obtain a Kähler potential $K = c\log(\phi^\dagger\phi) + \text{const}$, which is no longer linear.[5),6)] This procedure can be considered as a Kähler quotient method,[7)] and it is used to construct low energy effective theories of supersymmetric gauge theories.[8)] As an application of our exact quantum Legendre transform, we will show that *use of the classical equation of motion for V can be justified* in the path integral formalism. Moreover, as stated above, the Kähler potential can be generalized to $K = f(e^V\phi^\dagger\phi) - cV$ with an arbitrary function f. One of the main results of this paper is that we can show that this *arbitrariness disappears* after integrating out the vector superfield.

As an example, by combining these two auxiliary field methods, we construct a nonlinear sigma model, whose target space is the compact homogeneous Kähler manifold $SO(N)/SO(N-2)\times U(1)$, from a linear model. The construction of the hermitian symmetric spaces from linear models is discussed in a separate paper.[9)]

This paper is organized as follows. In §1 we prove a theorem on the quantum equivalence of auxiliary field methods in supersymmetric theories. In §2 we apply this theorem to supersymmetric nonlinear sigma models.

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*) Although, we discuss the Abelian model for simplicity in the Introduction, extension to the non-Abelian case is discussed below.

§2. Quantum Legendre transformation

Before proving the absence of quantum corrections, we summarize useful formulas for our proof. Vector superfields (real superfields) satisfy $V^\dagger = V$, and can be written in component fields as [16)]

$$\begin{aligned} V(x,\theta,\bar\theta) = C(x) &+ i\theta\chi(x) - i\bar\theta\bar\chi(x) \\ &+\frac{i}{2}\theta\theta[M(x)+iN(x)] - \frac{i}{2}\bar\theta\bar\theta[M(x)-iN(x)] \\ &-\theta\sigma^\mu\bar\theta v_\mu(x) + i\theta\theta\bar\theta\bar\lambda(x) - i\bar\theta\bar\theta\theta\lambda(x) + \frac{1}{2}\theta\theta\bar\theta\bar\theta D(x). \end{aligned} \tag{2·1}$$

The D-term of the product of two vector superfields is

$$\begin{aligned} V^iV^j|_{\theta\theta\bar\theta\bar\theta} = &\frac{1}{2}(C^iD^j + C^jD^i) - \frac{1}{2}(\chi^i\lambda^j + \bar\chi^i\bar\lambda^j) - \frac{1}{2}(\chi^j\lambda^i + \bar\chi^j\bar\lambda^i) \\ &+\frac{1}{2}(M^iM^j + N^iN^j - v^i\cdot v^j). \end{aligned} \tag{2·2}$$

Next an arbitrary function of n vector superfields, $f(V^1,\cdots,V^n)$, can be expanded in a Taylor series around $C=(C^1,\cdots,C^n)$ as

$$\begin{aligned} &f(V^1,\cdots,V^n) \\ &= f(C) + if_i(C)(\theta\chi^i - \bar\theta\bar\chi^i) \\ &+\frac{i}{2}\theta\theta\left[f_i(C)(M^i+iN^i) - \frac{i}{2}f_{ij}(C)\chi^i\chi^j\right] \\ &-\frac{i}{2}\bar\theta\bar\theta\left[f_i(C)(M^i-iN^i) + \frac{i}{2}f_{ij}(C)\bar\chi^i\bar\chi^j\right] \\ &-\theta\sigma^\mu\bar\theta\left[f_i(C)v^i{}_\mu + \frac{1}{2}f_{ij}(C)(\chi^{(i}\sigma_\mu\bar\chi^{j)})\right] \\ &+i\theta\theta\bar\theta\left[f_i(C)\bar\lambda^i - \frac{i}{2}f_{ij}(C)\bar\chi^{(i}(M+iN)^{j)} - \frac{1}{2}f_{ij}(C)\bar\sigma^\mu\chi^{(i}v^{j)}{}_\mu - \frac{1}{6}f_{ijk}(C)\bar\chi^{(i}\chi^j\chi^{k)}\right] \\ &-i\bar\theta\bar\theta\theta\left[f_i(C)\lambda^i + \frac{i}{2}f_{ij}(C)\chi^{(i}(M-iN)^{j)} + \frac{1}{2}f_{ij}(C)\sigma^\mu\bar\chi^{(i}v^{j)}{}_\mu - \frac{1}{6}f_{ijk}(C)\chi^{(i}\bar\chi^j\bar\chi^{k)}\right] \\ &+\frac{1}{2}\theta\theta\bar\theta\bar\theta\left[f_i(C)D^i - f_{ij}(C)(\chi^{(i}\lambda^{j)} + \bar\chi^{(i}\bar\lambda^{j)}) + \frac{1}{2}f_{ij}(C)(M^iM^j + N^iN^j - v^i\cdot v^j)\right. \\ &\qquad -\frac{1}{4}f_{ijk}(C)\{i(\chi^{(i}\chi^j - \bar\chi^{(i}\bar\chi^j)M^{k)} + (\chi^{(i}\chi^j + \bar\chi^{(i}\bar\chi^j)N^{k)} + 2v^{(i}{}_\mu(\chi^j\sigma^\mu\bar\chi^{k)})\} \\ &\qquad \left.+\frac{1}{8}f_{ijkl}(C)\chi^{(i}\chi^j\bar\chi^k\bar\chi^{l)}\right], \end{aligned} \tag{2·3}$$

where we have used the notation

$$A^{(i_1\cdots i_n)} = \frac{1}{n!}(A^{i_1\cdots i_n} + (\text{symmetrization})), \tag{2·4}$$

and the subscripts $i,j,\cdots$ denote differentiation with respect to $C^i, C^j, \cdots$.

Now we are ready to state the main result of this section regarding the absence of quantum corrections to the Legendre transform.

Theorem 1. (The quantum Legendre transformation.)
Let $\sigma^i(x,\theta,\bar\theta)$ and $\Phi_i(x,\theta,\bar\theta)$ $(i=1,\cdots,n)$ be vector superfields and W be a function of σ^i. Then,

$$\int[d\sigma]\exp\left[i\int d^4xd^4\theta\,(\sigma^i\Phi_i - W(\sigma^1,\cdots,\sigma^n))\right] = \exp\left[i\int d^4xd^4\theta\,U(\Phi)\right]. \quad (2\cdot5)$$

Here $U(\Phi)$ is defined by

$$U(\Phi) = \hat\sigma^i(\Phi)\Phi_i - W(\hat\sigma^1(\Phi),\cdots,\hat\sigma^n(\Phi)), \quad (2\cdot6)$$

where $\hat\sigma$ is a solution of the stationary equation

$$\frac{\partial}{\partial\sigma^i}(\sigma^j\Phi_j - W(\sigma^1,\cdots,\sigma^n))|_{\sigma^i=\hat\sigma^i} = \Phi_i - W_i(\hat\sigma^1,\cdots,\hat\sigma^n) = 0. \quad (2\cdot7)$$

(Proof) To show this, we calculate the path integral of σ on the left-hand side explicitly, and then compare with the right-hand side. By using Eqs. (2·2) and (2·3), the integrand of the left-hand side can be written as the exponential of

$$\begin{aligned}
&(\sigma^i\Phi_i - W(\sigma^1,\cdots,\sigma^n))|_{\theta\theta\bar\theta\bar\theta}\\
&= \frac{1}{2}D^i_\sigma(C_{\Phi i} - W_i(C_\sigma)) - \frac{1}{2}\lambda^i_\sigma(\chi_{\Phi i} - W_{ij}(C_\sigma)\chi^j_\sigma) - \frac{1}{2}\bar\lambda^i_\sigma(\bar\chi_{\Phi i} - W_{ij}(C_\sigma)\bar\chi^j_\sigma)\\
&\quad+\frac{1}{2}C^i_\sigma D_{\Phi i} - \frac{1}{2}(\chi^i_\sigma\lambda_{\Phi i} + \bar\chi^i_\sigma\bar\lambda_{\Phi i}) + \frac{1}{2}(M^i_\sigma M_{\Phi i} + N^i_\sigma N_{\Phi i} - v^i_\sigma\cdot v_{\Phi i})\\
&\quad-\frac{1}{4}W_{ij}(C_\sigma)(M^i_\sigma M^j_\sigma + N^i_\sigma N^j_\sigma - v^i_\sigma\cdot v^j_\sigma)\\
&\quad+\frac{1}{8}W_{ijk}(C_\sigma)\left[i(\chi^{(i}_\sigma\chi^j_\sigma - \bar\chi^{(i}_\sigma\bar\chi^{j}_\sigma)M^{k)}_\sigma + (\chi^{(i}_\sigma\chi^j_\sigma + \bar\chi^{(i}_\sigma\bar\chi^{j}_\sigma)N^{k)}_\sigma + 2v^{(i}_{\sigma\mu}(\chi^j_\sigma\sigma^\mu\bar\chi^{k)}_\sigma)\right]\\
&\quad-\frac{1}{16}W_{ijkl}(C_\sigma)\chi^{(i}_\sigma\chi^j_\sigma\bar\chi^k_\sigma\bar\chi^{l)}_\sigma. \qquad (2\cdot8)
\end{aligned}$$

The path integrals of D^i_σ, λ^i_σ and $\bar\lambda^i_\sigma$ give delta functions, $\delta(C_{\Phi i} - W_i(C_\sigma))$, $\delta(\chi_{\Phi i} - W_{ij}(C_\sigma)\chi^j_\sigma)$ and $\delta(\bar\chi_{\Phi i} - W_{ij}(C_\sigma)\bar\chi^j_\sigma)$, respectively. The equation $C_{\Phi i} - W_i(C_\sigma) = 0$ can be solved for C_σ uniquely: $C^i_\sigma = \hat C^i_\sigma(C_\Phi)$, on the assumption that $\det\boldsymbol{W}(C_\sigma) \neq 0$, where the Hesse matrix $\boldsymbol{W}$ is defined by

$$(\boldsymbol{W})_{ij}(C_\sigma) \stackrel{\text{def}}{=} W_{ij}(C_\sigma) = \frac{\partial^2 W}{\partial\sigma^i\partial\sigma^j}(C_\sigma). \quad (2\cdot9)$$

The fermionic fields χ_σ are given by

$$\hat\chi^i_\sigma = W(\hat C_\sigma)^{ij}\chi_{\Phi j}, \quad (2\cdot10)$$

where the inverse of the Hesse matrix is defined by

$$W^{ij} \stackrel{\text{def}}{=} (\boldsymbol{W}^{-1})^{ij}. \quad (2\cdot11)$$

The path integrals over C^i_σ, χ^i_σ and $\bar{\chi}^i_\sigma$ can be performed trivially because of the delta functions. These integrals leave the factors $|\det \boldsymbol{W}(\hat{C}_\sigma)|^{-1}$, $(\det \boldsymbol{W}(\hat{C}_\sigma))^2$ and $(\det \boldsymbol{W}(\hat{C}_\sigma))^2$, respectively. The remaining fields M^i_σ, N^i_σ and $v^i_{\sigma\mu}$ are quadratic:

$$\begin{aligned}
&-\frac{1}{4}W_{ij}\Big[(M_\sigma-\hat{M}_\sigma)^i(M_\sigma-\hat{M}_\sigma)^j+(N_\sigma-\hat{N}_\sigma)^i(N_\sigma-\hat{N}_\sigma)^j \\
&\qquad -(v_\sigma-\hat{v}_\sigma)^i\cdot(v_\sigma-\hat{v}_\sigma)^j\Big] \\
&+\frac{1}{4}W_{ij}\left[\hat{M}^i_\sigma\hat{M}^j_\sigma+\hat{N}^i_\sigma\hat{N}^j_\sigma-\hat{v}^i_\sigma\cdot\hat{v}^j_\sigma\right],
\end{aligned} \tag{2·12}$$

where we have defined

$$\begin{aligned}
\hat{M}^i_\sigma &= W^{ij}M_{\Phi j}+\frac{i}{4}W^{ij}W_{jkl}W^{km}W^{ln}(\chi_{\Phi m}\chi_{\Phi n}-\bar{\chi}_{\Phi m}\bar{\chi}_{\Phi n}), \\
\hat{N}^i_\sigma &= W^{ij}N_{\Phi j}+\frac{1}{4}W^{ij}W_{jkl}W^{km}W^{ln}(\chi_{\Phi m}\chi_{\Phi n}+\bar{\chi}_{\Phi m}\bar{\chi}_{\Phi n}), \\
\hat{v}^i_{\sigma\mu} &= W^{ij}v_{\Phi j\mu}-\frac{1}{2}W^{ij}W_{jkl}W^{km}W^{ln}(\chi_{\Phi(m}\sigma_\mu\bar{\chi}_{\Phi n)}).
\end{aligned} \tag{2·13}$$

Here, we have used the notation $W_i = W_i(\hat{C}_\sigma)$, etc. The path integrals over M^i_σ, N^i_σ and $v^i_{\sigma\mu}$ provide the factors $(\det \boldsymbol{W})^{-\frac{1}{2}}$, $(\det \boldsymbol{W})^{-\frac{1}{2}}$ and $[(\det \boldsymbol{W})^{-\frac{1}{2}}]^4$, respectively. Thus, the result of the path integral can be written as $A\exp(i\int d^4x\Gamma)$, where A and Γ are as follows:

$$A=\frac{\det \boldsymbol{W}}{|\det \boldsymbol{W}|}=\mathrm{sign}(\det \boldsymbol{W}), \tag{2·14}$$

$$\begin{aligned}
\Gamma &= \frac{1}{2}\hat{C}_\sigma(C_\Phi)^iD_{\Phi i}+\frac{1}{4}W^{ij}(M_{\Phi i}M_{\Phi j}+N_{\Phi i}N_{\Phi j}-v_{\Phi i}\cdot v_{\Phi j}) \\
&\quad -\frac{1}{2}W^{ij}(\chi_{\Phi(i}\lambda_{\Phi j)}+\bar{\chi}_{\Phi(i}\bar{\lambda}_{\Phi j)}) \\
&\quad +\frac{1}{8}W^{ij}W_{jkl}W^{km}W^{ln}[i(\chi_{\Phi m}\chi_{\Phi n}-\bar{\chi}_{\Phi m}\bar{\chi}_{\Phi n})M_{\Phi i}+(\chi_{\Phi m}\chi_{\Phi n}+\bar{\chi}_{\Phi m}\bar{\chi}_{\Phi n})N_{\Phi i} \\
&\qquad\qquad +2v_{\Phi i\mu}(\chi_{\Phi(m}\sigma^\mu\bar{\chi}_{\Phi n)})] \\
&\quad +\frac{1}{16}\left(3W^{ij}W_{ikl}W_{jmn}-W_{klmn}\right)W^{ko}W^{lp}W^{mq}W^{nr}\chi_{\Phi(o}\chi_{\Phi p}\bar{\chi}_{\Phi q}\bar{\chi}_{\Phi r)}.
\end{aligned} \tag{2·15}$$

When the correspondence between C_σ and C_Φ is not unique, we should choose the branch where $\det \boldsymbol{W}(C_\sigma)$ has a definite sign.

To prove the theorem, we calculate the D-term of $U(\Phi)$ by solving Eq. (2·7) explicitly, and compare with Γ in Eq. (2·15). Using Eq. (2·3), we can express Eq. (2·7) by components as

$$\begin{aligned}
0 &= \Phi_i-W_i(\sigma^1,\cdots,\sigma^n) \\
&= C_{\Phi i}-W_i(C_\sigma)+i\theta(\chi_{\Phi i}-W_{ij}\chi^j_\sigma)-i\bar{\theta}(\bar{\chi}_{\Phi i}-W_{ij}\bar{\chi}^j_\sigma) \\
&\quad +\frac{i}{2}\theta\theta\left[M_{\Phi i}+iN_{\Phi i}-W_{ij}(M^j_\sigma+iN^j_\sigma)+\frac{i}{2}W_{ijk}\chi^j_\sigma\chi^k_\sigma\right] \\
&\quad -\frac{i}{2}\bar{\theta}\bar{\theta}\left[M_{\Phi i}-iN_{\Phi i}-W_{ij}(M^j_\sigma-iN^j_\sigma)-\frac{i}{2}W_{ijk}\bar{\chi}^j_\sigma\bar{\chi}^k_\sigma\right]
\end{aligned}$$

$$
\begin{aligned}
&-\theta\sigma^\mu\bar\theta\left[v_{\Phi i\mu}-W_{ij}v^j_{\sigma\mu}-\frac{1}{2}W_{ijk}(\chi^{(j}_\sigma\sigma_\mu\bar\chi^{k)}_\sigma)\right]\\
&+i\theta\theta\bar\theta\Big[\bar\lambda_{\Phi i}-W_{ij}\bar\lambda^j_\sigma+\frac{i}{2}W_{ijk}\bar\chi^{(j}_\sigma(M_\sigma+iN_\sigma)^{k)}\\
&\quad+\frac{1}{2}W_{ijk}\bar\sigma^\mu\chi^{(j}_\sigma v^{k)}_{\sigma\mu}+\frac{1}{6}W_{ijkl}\bar\chi^{(j}_\sigma\chi^k_\sigma\chi^{l)}_\sigma\Big]\\
&-i\bar\theta\bar\theta\theta\Big[\lambda_{\Phi i}-W_{ij}\lambda^j_\sigma-\frac{i}{2}W_{ijk}\chi^{(j}_\sigma(M_\sigma-iN_\sigma)^{k)}\\
&\quad-\frac{1}{2}W_{ijk}\sigma^\mu\bar\chi^{(j}_\sigma v^{k)}_{\sigma\mu}+\frac{1}{6}W_{ijkl}\chi^{(j}_\sigma\bar\chi^k_\sigma\bar\chi^{l)}_\sigma\Big]\\
&+\frac{1}{2}\theta\theta\bar\theta\bar\theta\Big[D_{\Phi i}-W_{ij}(C)D^j_\sigma+W_{ijk}(C)(\chi^{(j}_\sigma\lambda^{k)}_\sigma+\bar\chi^{(j}_\sigma\bar\lambda^{k)}_\sigma)\\
&\quad-\frac{1}{2}W_{ijk}(C)(M^j_\sigma M^k_\sigma+N^j_\sigma N^k_\sigma-v^j_\sigma\cdot v^k_\sigma)\\
&\quad-\frac{1}{4}W_{ijkl}(C)\{i(\chi^{(j}_\sigma\chi^k_\sigma-\bar\chi^{(j}_\sigma\bar\chi^k_\sigma)M^{l)}_\sigma+(\chi^{(j}_\sigma\chi^k_\sigma+\bar\chi^{(j}_\sigma\bar\chi^k_\sigma)N^{l)}_\sigma+2v^{(j}_{\sigma\mu}(\chi^k_\sigma\sigma^\mu\bar\chi^{l)}_\sigma)\}\\
&\quad-\frac{1}{8}W_{ijklm}(C)\chi^{(j}_\sigma\chi^k_\sigma\bar\chi^l_\sigma\bar\chi^{m)}_\sigma\Big].
\end{aligned}
\tag{2·16}
$$

These equations can be solved by components as

$$
\begin{aligned}
&C_{\Phi i}=W_i(\hat C_\sigma),\\
&\hat\chi^i_\sigma=W^{ij}\chi_{\Phi j},\\
&\hat M^i_\sigma=W^{ij}M_{\Phi j}+\frac{i}{4}W^{ij}W_{jkl}W^{km}W^{ln}(\chi_{\Phi m}\chi_{\Phi n}-\bar\chi_{\Phi m}\bar\chi_{\Phi n}),\\
&\hat N^i_\sigma=W^{ij}N_{\Phi j}+\frac{1}{4}W^{ij}W_{jkl}W^{km}W^{ln}(\chi_{\Phi m}\chi_{\Phi n}+\bar\chi_{\Phi m}\bar\chi_{\Phi n}),\\
&\hat v^i_{\sigma\mu}=W^{ij}v_{\Phi j\mu}-\frac{1}{2}W^{ij}W_{jkl}W^{km}W^{ln}(\chi_{\Phi(m}\sigma_\mu\bar\chi_{\Phi n)}).
\end{aligned}
\tag{2·17}
$$

Here we have not written the solutions for $\hat\lambda_\sigma$ and $\hat D_\sigma$, since they disappear when the first and second equations are substituted into Eq. (2·8). The first equation can be solved implicitly by $\hat C_\sigma$. By substituting these equations into Eq. (2·8), we confirm that it coincides with Eq. (2·15). (Q.E.D.)

We give the chiral superfield version of Theorem 1 in Appendix A.

We can also prove the matrix version of Theorem 1, which will be applied to integration over the non-Abelian vector superfields in the next section.

Corollary 1. (The matrix version of the quantum Legendre transformation.) Let Σ and Ψ be matrix-valued vector superfields and let $W(\Sigma)$ be a scalar function of Σ.*) Then,

$$
\int[d\Sigma]\exp\left[i\int d^4xd^4\theta\,(\mathrm{tr}\,(\Sigma\Psi)-W(\Sigma))\right]=\exp\left[i\int d^4xd^4\theta\,U(\Psi)\right]. \tag{2·18}
$$

*) For example, it is a single trace of a function w, $W(\Sigma)=\mathrm{tr}\,w(\Sigma)$, but it need not be a single trace.

Here $U(\Psi)$ is defined as

$$U(\Psi) = \mathrm{tr}\,(\hat{\Sigma}(\Psi)\Psi) - W\left(\hat{\Sigma}(\Psi)\right), \tag{2·19}$$

where $\hat{\Sigma}$ is a solution of the stationary equation

$$\frac{\partial}{\partial \Sigma_{ij}}\left[\mathrm{tr}\,(\Sigma\Psi - W(\Sigma))\right]\Big|_{\Sigma=\hat{\Sigma}} = \Psi_{ij} - \left(\partial_\Sigma W(\hat{\Sigma})\right)_{ij} = 0. \tag{2·20}$$

(Proof) The proof is trivial, if we rewrite $\mathrm{tr}\,(\Sigma\Psi) = \Sigma_{ij}\Psi_{ji}$ and identify a pair of indices ij with an index in Theorem 1. (Q.E.D.)

If the scalar function $W(\Sigma)$ is written in a single trace as $W(\Sigma) = \mathrm{tr}\, w(\Sigma)$ with a function w, its derivative with respect to Σ can be written as $\partial_{\Sigma_{ij}} W(\Sigma) = w'(\Sigma)^T{}_{ij}$. Note that, as in Eq. (2·14), we must assume that the Hessian has a definite sign. However, this is automatically the case for the example in the next section.

We derive one more useful corollary of Theorem 1, which will be used in the proof of Theorem 2 in the next section.

Corollary 2.
Let σ^i $(i = 1, \cdots, n)$ be vector superfields and W be a function of σ^i. Then,

$$\int [d\sigma] \exp\left[i \int d^4x d^4\theta\; W(\sigma^1, \cdots, \sigma^n)\right] = 1 \tag{2·21}$$

when the Hessian $\boldsymbol{W}_{ij} = \partial_i \partial_j W(C_\sigma)$ is positive definite.
(Proof) By Theorem 1, the path integral of σ is given by the solution $\hat{\sigma}^i$ of the stationary equation, $W_i(\sigma^1, \cdots, \sigma^n)|_{\sigma^i=\hat{\sigma}^i} = 0$. By using Eq. (2·3), it can be written in components as

$$W_i(\hat{C}^1_\sigma, \cdots, \hat{C}^n_\sigma) = 0, \qquad \hat{\chi}^i_\sigma = \hat{M}^i_\sigma = \hat{N}^i_\sigma = \hat{v}^i_\sigma = \hat{\lambda}^i_\sigma = \hat{D}^i_\sigma = 0, \tag{2·22}$$

if the Hessian, $\det \boldsymbol{W}(\hat{C}_\sigma)$, is not zero. The solutions of the first equations $\hat{\sigma}^i = \hat{C}^i_\sigma$ are c-numbers. Therefore, the left-hand side is

$$\begin{aligned}(\mathrm{LHS}) &= \exp\left[i \int d^4x d^4\theta\; W(\hat{\sigma}^1, \cdots, \hat{\sigma}^n)\right] \\ &= \exp\left[i \int d^4x d^4\theta\; W(\hat{C}^1_\sigma, \cdots, \hat{C}^n_\sigma)\right] = 1\end{aligned} \tag{2·23}$$

when the Hessian is positive definite. (Q.E.D.)

In the remainder of this section, we prove that the path integral measure is invariant under a change of variables in supersymmetric theories.

Lemma 1. (The invariance of the measure.)
When two sets of n vector superfields are related by a nonsingular local transformation,

$$\sigma^i = f^i(V^1, \cdots, V^n), \quad (i = 1, \cdots, n) \tag{2·24}$$

the measure is invariant:

$$[d\sigma] = [dV]. \tag{2·25}$$

(Proof) The measures of the superfields are defined by their component fields as

$$[d\sigma] \stackrel{\text{def}}{=} \left[\prod_i d\sigma^i\right], \quad d\sigma^i \stackrel{\text{def}}{=} dC^i_\sigma d\chi^i_\sigma d\bar{\chi}^i_\sigma dM^i_\sigma dN^i_\sigma dv^i_{\sigma m} d\bar{\lambda}^i_\sigma d\lambda^i_\sigma dD^i_\sigma,$$
$$[dV] \stackrel{\text{def}}{=} \left[\prod_i dV^i\right], \quad dV^i \stackrel{\text{def}}{=} dC^i_V d\chi^i_V d\bar{\chi}^i_V dM^i_V dN^i_V dV^i_{Vm} d\bar{\lambda}^i_V d\lambda^i_V dD^i_V. \tag{2·26}$$

For convenience, we reorder them as $[d\sigma] = [\prod_i dC^i_\sigma \prod_i d\chi^i_\sigma \cdots]$, etc. From Eq. (2·3), the superfield equation, $\sigma^i = f^i(V)$, can be written in component fields as*)

$$\begin{aligned}
&C^i_\sigma = f^i(C_V), \quad \chi^i_\sigma = f^i{}_j(C_V)\chi^j_V, \quad \bar{\chi}^i_\sigma = f^i{}_j(C_V)\bar{\chi}^j_V,\\
&M^i_\sigma = f^i{}_j(C_V)M^j_V - \frac{i}{4}f^i{}_{jk}(C_V)(\chi^j_V\chi^k_V - \bar{\chi}^j_V\bar{\chi}^k_V),\\
&N^i_\sigma = f^i{}_j(C_V)N^j_V - \frac{1}{4}f^i{}_{jk}(C_V)(\chi^j_V\chi^k_V + \bar{\chi}^j_V\bar{\chi}^k_V),\\
&v^i_{\sigma\mu} = f^i{}_j(C_V)v^j_{V\mu} + \frac{1}{2}f^i{}_{jk}(C_V)(\chi^{(j}_V\sigma_\mu\bar{\chi}^{k)}_V), \quad \cdots.
\end{aligned} \tag{2·27}$$

The measures in the two sets of coordinates are related by

$$[d\sigma] = J[dV], \tag{2·28}$$

where J is the Jacobian, $J = \operatorname{sdet} M$. Here, sdet is a superdeterminant of a $16n \times 16n$ supermatrix M (8 bosonic and 8 fermionic components for each vector superfield), whose diagonal blocks are all $f^i{}_j(C_V)$, and whose off-diagonal part is nilpotent:

$$M = \mathbf{1}_{16} \otimes \boldsymbol{f}(C_V) + M', \tag{2·29}$$

where $(\boldsymbol{f})^i{}_j = f^i{}_j$ and M' is a nilpotent matrix. Then, J can be written as

$$J = \operatorname{sdet} M = \exp(\operatorname{str}\log M), \tag{2·30}$$

where str is a supertrace. This can be calculated as**)

$$\begin{aligned}
&\operatorname{sdet} M = \operatorname{sdet} A \cdot \operatorname{sdet} B,\\
&\log A = \log(\mathbf{1}_{16} \otimes \boldsymbol{f}(C_V)),\\
&\log B = \log\left(\mathbf{1}_{16n} + (\mathbf{1}_{16} \otimes \boldsymbol{f}^{-1})M'\right) = \sum_{m>0} \frac{(-1)^m}{m}\left((\mathbf{1}_{16} \otimes \boldsymbol{f}^{-1})M'\right)^m,
\end{aligned} \tag{2·31}$$

where $\boldsymbol{f}^{-1}$ is the inverse matrix of $\boldsymbol{f}$. Since the supertrace of $\log A$ is zero due to the cancellation of bosonic and fermionic contributions $(8 - 8 = 0)$, $\operatorname{sdet} A = 1$. Because M' is nilpotent, $\log B$ is also nilpotent, and hence $\operatorname{sdet} B = 1$ and $J = 1$. (Q.E.D.)

The invariance of the measure also holds for matrix-valued superfields.

*) Note that the ranges of the variables remain unchanged, except for C_σ and C_V, provided that $f^i{}_j(C_V) \neq 0$.

**) If we write $M = AB$, then from the equation $\log M = \log(AB) = \log A + \log B + \frac{1}{2}[\log A, \log B] + \cdots$, we obtain the formula $\operatorname{sdet} M = \operatorname{sdet} A \cdot \operatorname{sdet} B$.

Lemma 2. (The invariance of the measure for matrix-valued superfields.)
Let Σ and V be matrix valued vector superfields related by a local transformation $\Sigma = f(V)$. Then

$$[d\Sigma] = [dV], \tag{2·32}$$

where the measures are defined by $[d\Sigma] = \prod_{ij} d\Sigma_{ij}$ and $[dV] = \prod_{ij} dV_{ij}$.
(Proof) The proof is trivial, since $\Sigma_{ij} = f^{ij}(V_{11}, \cdots, V_{NN})$, where f^{ij} are functions of $V_{11}, \cdots, V_{NN}$. (Q.E.D.)

§3. Applications to nonlinear sigma models

In this section, we apply our results to show the equivalence of two Lagrangians with and without auxiliary fields. Firstly, we apply the theorems obtained in the last section to non-Abelian gauge theories, where the vector superfields are auxiliary fields. Consider the $N \times M$ matrix valued chiral superfields Φ belonging to the fundamental representation of global symmetry $U(N)$ and the anti-fundamental representation of the gauge group $U(M)$. The action of the global and gauge symmetries are as follows:

$$\Phi \to g_L \Phi {g_R}^{-1}, \tag{3·1}$$

where g_L is an $N \times N$ matrix of $U(N)$ and g_R is an $M \times M$ matrix of $U(M)$. The gauge invariant Lagrangian of Φ interacting with the vector superfield $V = V^A T_A$, where T_A is the Lie algebra of $U(M)$, reads

$$\mathcal{L} = \int d^4\theta K_0(\Phi, \Phi^\dagger, V) = \int d^4\theta \left(\mathrm{tr}\,(\Phi^\dagger \Phi e^V) - c\,\mathrm{tr}\, V \right). \tag{3·2}$$

Here c is the Fayet-Iliopoulos (FI) parameter.

As described in the Introduction, we can obtain the Kähler potential of the nonlinear sigma model when we eliminate V with its equation of motion. As an application of the theorem obtained in the last section, we show that this procedure is also justified at the quantum level. We obtain the next corollary from Corollary 1.

Corollary 3. (Integrating out the non-Abelian vector superfields.)
Let Φ be an $N \times M$ matrix-valued chiral superfield and V be a matrix-valued vector superfield $V = V^A T_A$, where T_A is the $M \times M$ matrix of a Lie algebra. Then

$$\int [dV] \exp\left[i \int d^4x d^4\theta \left(\mathrm{tr}\,(\Phi^\dagger \Phi e^V) - c\,\mathrm{tr}\, V \right) \right] = \exp\left[i \int d^4x d^4\theta\, c\,\mathrm{tr}\, \log(\Phi^\dagger \Phi) \right]. \tag{3·3}$$

(Proof) In Corollary 1, we replace Σ and Ψ by e^V and $\Phi^\dagger \Phi$, respectively. Then the left-hand side can be calculated as

$$\begin{aligned} (\mathrm{LHS}) &= \int [d\Sigma] \exp\left[i \int d^4x d^4\theta \left(\mathrm{tr}\,(\Sigma \Psi) - c\,\mathrm{tr}\, \log \Sigma \right) \right] \\ &= \exp\left[i \int d^4x d^4\theta\, U(\Psi) \right] = (\mathrm{RHS}), \end{aligned} \tag{3·4}$$

where U has been defined as $U(\Psi) = \mathrm{tr}\,(\hat{\Sigma}(\Psi)\Psi) - c\,\mathrm{tr}\,\log\hat{\Sigma}(\Psi)$ and $\hat{\Sigma}(\Psi) = c\Psi^{-1}$. We have used Lemma 2 in the first equality and Corollary 1 in the second equality. The constant term disappears under integration over θ: $\int d^4\theta(iNc - iNc\log c) = 0$. It is proven in Appendix B that the Hessian $\det \boldsymbol{W}$ in Eq. (2·14) has a definite sign. (Q.E.D.)

Secondly, we apply the theorems to show the uniqueness of the effective Lagrangian of the supersymmetric nonlinear sigma models on compact Kähler manifolds.

When a global symmetry G is spontaneously broken, there appear Nambu-Goldstone (NG) bosons corresponding to the broken generators. Since a chiral superfield has two real scalar fields, the partner of a NG-boson may not be a NG-boson. If this is the case, the partner of the NG-boson is called a quasi-Nambu-Goldstone (QNG) boson.[10] When there is no gauge symmetry, it is known that there must be at least one QNG boson,[11] and the Kähler potential of the effective Lagrangian can be written as an arbitrary function of the G invariants.[10),2),11] QNG bosons correspond to the *non-compact* directions of the target space and introduce an arbitrariness into the Kähler potential. On the other hand, the Kähler potential has no arbitrariness[2),12] if the target space is *compact*. Our task in the remainder of this section is to show that the arbitrariness disappears in the process of eliminating the non-compact directions. We introduce gauge fields to eliminate the QNG-bosons corresponding to non-compact directions. When matrix-valued chiral superfields Φ acquire the vacuum expectation value, the most general effective Kähler potential can be written as[2),11]

$$K_0(\Phi, \Phi^\dagger) = f(\mathrm{tr}\,(\Phi^\dagger\Phi)), \tag{3·5}$$

where f is an *arbitrary* function introduced by the QNG bosons.*) When we introduce gauge fields to eliminate non-compact directions, the gauge invariant Kähler potential reads

$$K_0 = f(\mathrm{tr}\,(\Phi^\dagger\Phi e^V)) - c\,\mathrm{tr}\,V. \tag{3·6}$$

The arbitrariness of the Kähler potential comes from the existence of the QNG bosons. Hence, if the gauge fields absorb all the QNG bosons, the target manifold becomes compact,[12] and the arbitrariness disappears after integrating out the vector superfield. We show that this actually is the case.**)

Firstly, we generalize Corollary 1 to the case in which an arbitrary function appears.

Theorem 2.
Let Σ and Ψ be $N \times N$ matrix-valued vector superfields. Then the arbitrary function

*) The low-energy interactions of NG and QNG bosons were obtained as low energy theorems.[13] Low-energy scattering amplitudes of NG bosons coincide with those of non-supersymmetric theories, despite the existence of the arbitrariness. Those of QNG bosons coincide with those of corresponding NG bosons. Moreover, the arbitrariness appears in the interactions of NG and QNG bosons.

**) It is known that, at the classical level, this arbitrariness must disappear when V is eliminated by using its equation of motion.[14]

f disappears for $W(\Sigma) = c\,\mathrm{tr}\,\log\Sigma \stackrel{\mathrm{def}}{=} c\,\mathrm{tr}\,w(\Sigma)$:

$$\int [d\Sigma] \exp\left[i\int d^4x d^4\theta \Big(f(\mathrm{tr}\,(\Sigma\Psi)) - W(\Sigma)\Big)\right]$$
$$= \int [d\Sigma] \exp\left[i\int d^4x d^4\theta \left(\mathrm{tr}\,(\Sigma\Psi) - W(\Sigma)\right)\right]. \tag{3·7}$$

(Proof) Let us introduce the Legendre transform $\tilde{W}$ of the arbitrary function f:

$$f(\phi) = \rho\phi - \tilde{W}(\rho), \qquad \phi - \tilde{W}'(\rho) = 0. \tag{3·8}$$

Then, the arbitrary function of $\mathrm{tr}\,(\Sigma\Psi)$ can be linearized as

$$\exp\left[i\int d^4x d^4\theta f(\mathrm{tr}\,(\Sigma\Psi))\right]$$
$$= \int [d\rho] \exp\left[i\int d^4x d^4\theta (\rho\,\mathrm{tr}\,(\Sigma\Psi) - \tilde{W}(\rho))\right].$$

By substituting this expression into the left-hand side and introducing the rescaled vector superfields $\Sigma' = \rho\Sigma$, we obtain*)

$$\begin{aligned}(\mathrm{LHS}) &= \int [d\Sigma d\rho] \exp\left[i\int d^4x d^4\theta (\rho\,\mathrm{tr}\,(\Sigma\Psi) - \tilde{W}(\rho) - c\,\mathrm{tr}\,w(\Sigma))\right] \\ &= \int [d\Sigma' d\rho] \exp\left[i\int d^4x d^4\theta \left(\mathrm{tr}\,(\Sigma'\Psi) - \tilde{W}(\rho) - c\,\mathrm{tr}\,w\left(\rho^{-1}\Sigma'\right)\right)\right] \\ &= \int [d\Sigma'] \exp\left[i\int d^4x d^4\theta \left(\mathrm{tr}\,(\Sigma'\Psi) - c\,\mathrm{tr}\,w(\Sigma')\right)\right] = (\mathrm{RHS}),\end{aligned} \tag{3·9}$$

where in the third equality, the decoupled ρ integral is evaluated by using Corollary 2:

$$\int [d\rho] \exp\left[-i\int d^4x d^4\theta \left(\tilde{W}(\rho) - cNw(\rho)\right)\right] = 1. \tag{3·10}$$

(Q.E.D.)

Note that the relation $W(\Sigma) = c\log\Sigma$ is essential to decouple the ρ integration, and this theorem cannot be generalized to the case of arbitrary W. However, it is sufficient for our purpose to show the following corollary.

Corollary 4. (The uniqueness of the seed metric.)
Consider the situation considered in Corollary 3. Then,

$$\int [dV] \exp\left[i\int d^4x d^4\theta \left(f(\mathrm{tr}\,(\Phi^\dagger\Phi e^V)) - c\,\mathrm{tr}\,V\right)\right]$$
$$= \int [dV] \exp\left[i\int d^4x d^4\theta \left(\mathrm{tr}\,(\Phi^\dagger\Phi e^V) - c\,\mathrm{tr}\,V\right)\right]. \tag{3·11}$$

(Proof) This result directly follows from Theorem 2 with the identification of $\Sigma = e^V$ and $\Psi = \Phi^\dagger\Phi$. The triviality of the Jacobian follows from Lemma 2. (Q.E.D.)

*) Note that the Jacobian is trivial by Lemma 1.

This corollary shows that the arbitrariness in effective theories without a gauge interaction like Eq. (3·5), disappears through the introduction of vector superfields like Eq. (3·6). Hence, we obtain the same Kähler potentials by Corollary 3. Since the local gauge symmetry eliminates the degrees of freedom in non-compact directions, the obtained manifold becomes compact.

Before closing this section, we give a simple example: $SO(N)/SO(N-2) \times U(1)$.[9] Consider chiral superfields $\vec{\phi}$ belonging to the vector representation of $SO(N)$. We impose the $O(N)$ invariant constraint $\vec{\phi}^2 = 0$ to define a manifold. This manifold is non-compact, since the defining condition is invariant under the scale transformation $\vec{\phi} \to \lambda\vec{\phi}$. We can eliminate this scale invariance by introducing a $U(1)$ gauge field for the local scale (complex phase) symmetry. The gauged Lagrangian is

$$\mathcal{L} = \int d^4\theta \left(f(e^V \vec{\phi}^\dagger \vec{\phi}) - cV \right) + \left(\int d^2\theta \, g\phi_0 \vec{\phi}^2 + \text{c.c.} \right), \tag{3·12}$$

where V and ϕ_0 are auxiliary vector and chiral superfields. We use a rather unconventional $SO(N)$ basis such that the invariant metric becomes

$$J = \begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{1}_{N-2} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix}. \tag{3·13}$$

Then the ϕ_0 integration gives the F-term constraint

$$\vec{\phi}^2 = \vec{\phi}^T J \vec{\phi} = 2ab + \varphi^2 = 0, \tag{3·14}$$

where we have put $\vec{\phi}^T = (a, \varphi^i, b)$. This equation can be easily solved as $\vec{\phi}^T = (a, \varphi^i, -\varphi^2/2a)$. Our theory has a gauge invariance which has the degree of freedom of a chiral superfield. To fix the gauge, we set $a = 1$, giving $\vec{\phi}^T = (1, \varphi^i, -\varphi^2/2)$. By Corollaries 4 and 3, the integration of V gives the Kähler potential,

$$K = c\log(\vec{\phi}^\dagger \vec{\phi}) = c\log\left(1 + |\varphi|^2 + \frac{1}{4}\varphi^{\dagger 2}\varphi^2\right). \tag{3·15}$$

This is the Kähler potential of a compact homogeneous Kähler manifold, $Q_{N-2}(\boldsymbol{C}) = SO(N)/SO(N-1) \times U(1)$,[12),4)] which is called the "quadratic surface".[15)] If we drop the F-term constraint, this becomes the Kähler potential for the Fubini-Study metric of $\boldsymbol{C}P^{N-1}$. Hence, $Q_{N-2}(\boldsymbol{C})$ is holomorphicaly embedded into $\boldsymbol{C}P^{N-1}$ by the F-term constraint Eq. (3·14).

Generalization to a broader class, namely the hermitian symmetric spaces, is discussed in a separate paper.[9)]

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Appendix A

Chiral Superfield Version of Theorem 1

In this appendix, we give the chiral superfield version of Theorem 1 without any proof.

Theorem 3. (Chiral superfield version.)

Let φ^i and ϕ_i $(i = 1, \cdots, n)$ be chiral superfields and W be a function of φ^i. Then,

$$\int [d\varphi] \exp i \left[\int d^4x d^2\theta \, (\varphi^i \phi_i - W(\varphi^1, \cdots, \varphi^n)) + (\text{c.c.}) \right]$$
$$= \exp i \left[\int d^4x d^2\theta \, U(\phi) + (\text{c.c.}) \right]. \tag{A·1}$$

Here $U(\phi)$ is defined as

$$U(\phi) = \hat{\varphi}^i(\phi)\phi_i - W(\hat{\varphi}^1(\phi), \cdots, \hat{\varphi}^n(\phi)), \tag{A·2}$$

where $\hat{\varphi}$ is a solution of the stationary equation

$$\frac{\partial}{\partial \varphi^i}(\varphi^j \phi_j - W(\varphi^1, \cdots, \varphi^n))|_{\varphi^i = \hat{\varphi}^i} = \phi_i - W_i(\hat{\varphi}^1, \cdots, \hat{\varphi}^n) = 0. \tag{A·3}$$

Appendix B

Definiteness of the Sign of the Hessian

Let Σ be an N by N matrix. In this appendix, for the case $W(\Sigma) = c \, \mathrm{tr} \log \Sigma$, we calculate the Hessian and show that its sign is definite. The Hesse matrix consists of derivatives of W with respect to two Σ_{ij} variables:

$$\boldsymbol{W}_{ij,kl}(C_\Sigma) = \frac{\partial^2 W(C_\Sigma)}{\partial \Sigma_{ij} \partial \Sigma_{kl}} = c \frac{\partial {\Sigma^{-1}}_{ji}}{\partial \Sigma_{kl}}. \tag{B·1}$$

By differentiating an identity $\delta_{ij} = \Sigma_{ik} {\Sigma^{-1}}_{kj}$, we obtain the formula

$$\frac{\partial {\Sigma^{-1}}_{ji}}{\partial \Sigma_{kl}} = -{\Sigma^{-1}}_{jk} {\Sigma^{-1}}_{li}. \tag{B·2}$$

Then the Hesse matrix can be calculated as

$$\boldsymbol{W}_{ij,kl}(C_\Sigma) = -c \Sigma^{-1}_{jk} \Sigma^{-1}_{li}. \tag{B·3}$$

Hence we can calculate the Hessian as*)

$$\det_{N^2 \times N^2} \boldsymbol{W} = (-c)^{N^2} (\det \Sigma^{-1})^{2N^2}. \tag{B·4}$$

The sign of this is definite, as used in the proof of Corollary 3.

*) When a matrix M can be written as a tensor product of two N by N matrices u and v as $M = u \otimes v$, its determinant is $\det M = (\det u)^{N^2} (\det v)^{N^2}$.

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