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Schwinger-Type Boson Realization for Three Sub-Algebras of the $su(4)$ Algebra

—— *The $so(5)$, the $so(4)$ and the $su(2) \otimes su(2)$ Algebra* ——

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The Schwinger-type boson realizations of the $so(5)$, $so(4)$ and $su(2) \otimes su(2)$ algebras are discussed in relation to the $su(4)$ algebra. The basic idea comes from a general form of the Schwinger-type boson representation for the $su(n)$ algebra, which was recently proposed by two of the present authors (M.Y. & A.K.), with da Providência. For each algebra, the orthogonal set to obtain a full irreducible representation is derived.

§1. Introduction and preliminaries

Since the boson expansion theories were proposed by Belyaev and Zelevinsky in 1962 [1)] and independently by Marumori, Yamamura (one of the present authors) and Tokunaga in 1964, [2)] the boson realization of Lie algebra has played a central role in the study of nuclear many-body problems, and in particular, in the field of collective dynamics. There are various studies related to this boson realization that are given as references in the review article by Klein and Marshalek, [3)] and more recent works are summarized in Ref. 4). For example, the Holstein-Primakoff-type boson realization for the $su(2)$ algebra is the simplest and the most common. In addition to the $su(2)$ algebra, in Ref. 3), the boson realization of the $so(5)$ algebra is discussed in detail. In the present, early stage of the development as a theory, in relation to the treatment of the charge-independent pairing correlation, this algebra has been considered in several studies. For example, one of the present authors (M.Y.), with Eichler, has addressed this problem. [5)] In this pairing theory, ten operators, which are expressed in term of the bilinear form for the neutron and proton operators, compose a Lie algebra, and they play an essential role. The definitions of these operators are given as follows:

$$\begin{aligned} \dot{D}^*_+ &= (1/2)\sum_m (-)^{j-m}\hat{n}^*_{+m}\hat{n}^*_{-m}, \\ \dot{D}_+ &= (1/2)\sum_m (-)^{j-m}\hat{n}_{-m}\hat{n}_{+m}, \\ \dot{D}^*_0 &= (1/2)\sum_m (-)^{j-m}\hat{n}^*_{+m}\hat{p}^*_{-m}, \end{aligned} \tag{1·1a}$$

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$$\dot{D}_0 = (1/2)\sum_m (-)^{j-m}\hat{p}_{-m}\hat{n}_{+m}, \tag{1·1b}$$

$$\dot{D}^*_- = (1/2)\sum_m (-)^{j-m}\hat{p}^*_{+m}\hat{p}^*_{-m},$$

$$\dot{D}_- = (1/2)\sum_m (-)^{j-m}\hat{p}_{-m}\hat{p}_{+m}, \tag{1·1c}$$

$$\dot{I}_+ = \sum_m \hat{n}^*_m\hat{p}_m,$$

$$\dot{I}_0 = (1/2)\sum_m (\hat{n}^*_m\hat{n}_m - \hat{p}^*_m\hat{p}_m),$$

$$\dot{I}_- = \sum_m \hat{p}^*_m\hat{n}_m, \tag{1·2}$$

$$\dot{M}_0 = (1/2)\sum_m (\hat{n}^*_m\hat{n}_m + \hat{p}^*_m\hat{p}_m) - \Omega. \quad (\Omega = (2j+1)/2) \tag{1·3a}$$

In the above operators, $(\hat{n}_m, \hat{n}^*_m)$ and $(\hat{p}_m, \hat{p}^*_m)$ denote neutron and proton operators in the $j-j$ coupling shell model orbit (j, m), respectively. The set $\{\dot{D}^*_{\pm,0}, \dot{D}_{\pm,0}, \dot{I}_{\pm,0}, \dot{M}_0\}$ composes the $so(5)$ algebra. The charge-independent pairing theory starts with the above ten operators. The operators $\dot{D}^*_{\pm,0}$ and $\dot{D}_{\pm,0}$ denote the charge-independent nucleon pairs, and the operators $\dot{I}_{\pm,0}$ are the iso-spin operators. The operator $\dot{M}_0$ is related to the total nucleon number. At the present stage, it may be recognized that the above-stated pairing problem has been regarded as classical in nuclear theory.

However, recent studies on high temperature superconductivity have forced us to find a new understanding of this algebra,[6] and its importance seems to be revived. Further, an interesting study concerning this problem has been proposed.[7] It is a study based on the $su(4)$ algebra. If we follow the basic idea of Ref. 7), we can add five operators to the ten given in the relations (1·1)–(1·3a) in the framework of the present fermion space. The five are defined as follows:

$$\dot{K}_+ = \sum_m (m/|m|)\hat{n}^*_m\hat{p}_m,$$

$$\dot{K}_0 = (1/2)\sum_m (m/|m|)(\hat{n}^*_m\hat{n}_m - \hat{p}^*_m\hat{p}_m),$$

$$\dot{K}_- = \sum_m (m/|m|)\hat{p}^*_m\hat{n}_m, \tag{1·4}$$

$$\dot{M}_+ = \sum_m (-)^{j-|m|}\hat{n}^*_{+m}\hat{p}^*_{-m},$$

$$\dot{M}_- = \sum_m (-)^{j-|m|}\hat{p}_{-m}\hat{n}_{+m}. \tag{1·3b}$$

It is interesting to note that the set $\{\dot{D}^*_{\pm,0}, \dot{D}_{\pm,0}, \dot{I}_{\pm,0}, \dot{K}_{\pm,0}, \dot{M}_{\pm,0}\}$ composes the $su(4)$ algebra, and the two sets $\{\dot{I}_{\pm,0}, \dot{K}_{\pm,0}\}$ and $\{\dot{I}_{\pm,0}, \dot{M}_{\pm,0}\}$ form the $so(4)$ and $su(2)\otimes su(2)$ algebras, respectively. The above implies that the $su(4)$ algebra has three sub-algebras. Clearly, the concrete expression for the $su(4)$ generators given in

Ref. 7) is different from that above, constructed for the convenience of the charge-independent pairing theory. According to a statement given in Ref. 7), the $so(5)$, $so(4)$ and $su(2) \otimes su(2)$ algebras correspond to a unification of superconductivity and antiferromagnetism, antiferromagnetic order, and D-wave pairing condensate, respectively. Further, Ref. 7) stresses that the above three appear as the sub-algebras of the $su(4)$ algbera, which provide a three-phase microscopic model of high temperature superconductivity and provide for us a clear understanding of the role played by the $so(5)$ algebra. This is a quite interesting assertion.

However, the analysis in Ref. 7) was performed with a special irreducible representation. The states treated in Ref. 7) consist only of the type $(\dot{D}^*_+)^{n_+}(\dot{D}^*_0)^{n_0}(\dot{D}_-)^{n_-}(\dot{M}_+)^m|\phi_0)$. The state $|\phi_0)$ denotes the vacuum for the nucleon $(\hat{n}_m|\phi_0) = \hat{p}_m|\phi_0) = 0)$. Therefore, in order to check the reliability of the results, we must perform the analysis with more general types of states. For this purpose, the idea of the boson realization of Lie algebra may be powerful, and a possible general form for the Schwinger-type boson representation of the $su(n)$ algebra may be useful. This was recently proposed by the present authors (M.Y. & A.K.), together with da Providência.[8] Hereafter, that work will be referred to as (I). In (I), the Schwinger-type boson representation of the $su(4)$ algebra is constructed in the frame with ($M = 3$ and arbitrary N). The case $M = 3$, $N = 0$ consists of four kinds of bosons. Then, the $so(4)$ and $su(2) \otimes su(2)$ algebras are easily obtained, because both can be represented in terms of four kinds of bosons. However, in order to obtain the full irreducible representation for the $so(5)$ algebra, at least six kinds of bosons must be introduced. This implies that the full representation of the $so(5)$ algebra cannot be obtained in the frame with $M = 3$, $N = 0$. However, the case $M = 3$, $N = 1$ consists of eight kinds of bosons, and thus in this case, it may be possible.

The main aim of this paper is to present a boson representation of the $so(5)$, $so(4)$ and $su(2) \otimes su(2)$ algebras as three sub-algebras of the $su(4)$ algebra. In (I), the generators of the su(4) algebra are expressed as S^i, S_i, S^i_j, with $i, j = 1, 2, 3$. In the present paper, after an appropriate rearrangement, we can define the operators $\hat{D}^*_{\pm,0}$, $\hat{D}_{\pm,0}$, $\hat{I}_{\pm,0}$, $\hat{K}_{\pm,0}$, $\hat{M}_{\pm,0}$, which corresond to the nucleon pair operators defined in the relations (1·1)–(1·4). The sets $\{\hat{D}^*_{\pm,0}, \hat{D}_{\pm,0}, \hat{I}_{\pm,0}, \hat{M}_0\}$, $\{\hat{I}_{\pm,0}, \hat{K}_{\pm,0}\}$ and $\{\hat{I}_{\pm,0}, \hat{M}_{\pm,0}\}$ compose the $so(5)$, $so(4)$ and $su(2) \otimes su(2)$ algebras, respectively. The Casimir operators enable us to construct the orthogonal set for each algebra. Concerning the construction of the orthogonal set, a method for obtaining an orthogonal set of a system with eight kinds of bosons, which hereafter is referred to as (II), is useful.[9] For instance, in the treatment of the $so(5)$ algebra, we can find two additional $su(2)$ and $su(1,1)$ algebras, which, including with the generators of the $so(5)$ algebra, commute with one another. The additional $su(2)$ algebra plays the role of the reduced iso-spin. The orthogonal set obtained in the above scheme is specified completely by the eight quantum numbers. Among these quantum numbers, six are used for the $so(5)$ algebra, and the remaining two do not have any influence on the representation of the $so(5)$ algebra. In our present frame, it may be possible to find a state that corresponds to the nucleon vacuum $|\phi_0)$ for each algebra. However, it may be impossible to find a state, such as $|\phi_0)$, that is common to the three algebras.

In this sense, for the purpose of finding a connection with the nucleon pair system, our present framework is applicable only to each algebra, and we cannot treat the $su(4)$ algebra in the case $M = 3$, $N = 1$ in relation to the nucleon pair system. For this purpose, we must extend the frame to the case $M = 3$, $N = 2$. This case will be considered in a subsequent paper.

In the next section, after giving a possible form of the $su(4)$ generators, the case $M = 3$, $N = 1$ is formulated. Section 3 is the most important part of this paper, in which the $so(5)$ algebra is discussed. Section 4 is devoted to formulating the $so(4)$ and $su(2) \otimes su(2)$ algebras. The method developed in (II) plays a central role in §§3 and 4. In §5, the connection with the nucleon pair system is discussed, and from this discussion, it is concluded that in the case of the $su(4)$ algebra, extension to the case $M = 3$, $N = 2$ is inevitable.

§2. Basic framework of the $su(4)$ algebra and its sub-algebras

It is convenient for the treatment of the $su(4)$ algebra to start from the $u(4)$ algebra. Following (I), we denote the $u(4)$ generators as $\hat{E}^i$, $\hat{E}_i$, $\hat{E}_i^j$, $\hat{E}$, with $i, j = 1, 2, 3$. Various relations are listed in Eqs. (I·2·3) and (I·2·4). The $su(4)$ generators denoted as $\hat{S}^i$, $\hat{S}_i$, $\hat{S}_i^j$, with $i, j = 1, 2, 3$, are given as

$$\hat{S}^i = \hat{E}^i, \quad \hat{S}_i = \hat{E}_i, \quad \hat{S}_i^j = \hat{E}_i^j + \delta_{ij}\hat{E}. \tag{2·1}$$

Further, with its properties, the operator S is introduced:

$$\hat{S} = \sum_{i=1}^{3} E_i^i - \hat{E}, \tag{2·2}$$

$$[\hat{S}, \hat{S}^i] = [\hat{S}, \hat{S}_i] = [\hat{S}, \hat{S}_i^j] = 0. \tag{2·2a}$$

Next, we define the operators $\hat{D}^*_{\pm,0}$, $\hat{D}_{\pm,0}$, $\hat{I}_{\pm,0}$, $\hat{K}_{\pm,0}$, $\hat{M}_{\pm,0}$:

$$\begin{aligned} &\hat{D}^*_+ = \hat{S}^3, \quad \hat{D}^*_- = \hat{S}^1_2, \quad \hat{D}^*_0 = (\hat{S}^2 - \hat{S}^1_3)/2, \\ &\hat{D}_+ = \hat{S}_3, \quad \hat{D}_- = \hat{S}^2_1, \quad \hat{D}_0 = (\hat{S}_2 - \hat{S}^3_1)/2, \end{aligned} \tag{2·3}$$

$$\begin{aligned} \hat{I}_+ &= \hat{S}^1 + \hat{S}^2_3, \quad \hat{I}_- = \hat{S}_1 + \hat{S}^3_2, \\ \hat{I}_0 &= (\hat{S}^1_1 - \hat{S}^2_2 + \hat{S}^3_3)/2, \end{aligned} \tag{2·4a}$$

$$\begin{aligned} \hat{K}_+ &= \hat{S}^1 - \hat{S}^2_3, \quad \hat{K}_- = \hat{S}_1 - \hat{S}^3_2, \\ \hat{K}_0 &= (\hat{S}^1_1 + \hat{S}^2_2 - \hat{S}^3_3)/2, \end{aligned} \tag{2·4b}$$

$$\hat{M}_+ = \hat{S}^2 + \hat{S}^1_3, \quad \hat{M}_- = \hat{S}_2 + \hat{S}^3_1, \tag{2·5}$$

$$\hat{M}_0 = (-\hat{S}^1_1 + \hat{S}^2_2 + \hat{S}^3_3)/2. \tag{2·6}$$

The set $\{\hat{I}_{\pm,0}\}$ forms the $su(2)$ algebra, and the three sets $\{\hat{D}^*_{\pm,0}\}$, $\{\hat{D}_{\pm,0}\}$ and $\{\hat{K}_{\pm,0}\}$ are vectors with respect to $\{\hat{I}_{\pm,0}\}$. The set $\{\hat{M}_{\pm,0}\}$ also forms the $su(2)$ algebra, and each of the three components is a scalar with respect to $\{\hat{I}_{\pm,0}\}$. It should be noted

that the following three sets compose the $so(5)$, $so(4)$ and $su(2)\otimes su(2)$ algebras, respectively:

$$so(5): \quad \{\hat{D}^*_{\pm,0}, \hat{D}_{\pm,0}, \hat{I}_{\pm,0}, \hat{M}_0\}, \tag{2·7}$$

$$so(4): \quad \{\hat{I}_{\pm,0}, \hat{K}_{\pm,0}\}, \tag{2·8}$$

$$su(2)\otimes su(2): \quad \{\hat{I}_{\pm,0}, \hat{M}_{\pm,0}\}. \tag{2·9}$$

The above three algebras are sub-algebras of the $su(4)$ algebra. Inversely, the set $\{\hat{S}^i, \hat{S}_i, \hat{S}^j_i\}$ can be expressed in terms of $\{\hat{D}^*_{\pm,0}, \hat{D}_{\pm,0}, \hat{I}_{\pm,0}, \hat{K}_{\pm,0}, \hat{M}_{\pm,0}\}$ in the following form:

$$\begin{aligned}
&\hat{S}^1 = (\hat{I}_+ + \hat{K}_+)/2, \quad \hat{S}^2 = \hat{M}_+/2 + \hat{D}^*_0, \quad \hat{S}^3 = \hat{D}^*_+,\\
&\hat{S}_1 = (\hat{I}_- + \hat{K}_-)/2, \quad \hat{S}_2 = \hat{M}_-/2 + \hat{D}_0, \quad \hat{S}_3 = \hat{D}_+,\\
&\hat{S}^1_1 = \hat{I}_0 + \hat{K}_0, \quad \hat{S}^2_2 = \hat{K}_0 + \hat{M}_0, \quad \hat{S}^3_3 = \hat{M}_0 + \hat{I}_0,\\
&\hat{S}^1_2 = \hat{D}^*_-, \quad \hat{S}^2_3 = (\hat{I}_+ - \hat{K}_+)/2, \quad \hat{S}^3_1 = \hat{M}_-/2 - \hat{D}_0,\\
&\hat{S}^2_1 = \hat{D}_-, \quad \hat{S}^3_2 = (\hat{I}_- - \hat{K}_-)/2, \quad \hat{S}^1_3 = \hat{M}_+/2 - \hat{D}^*_0.
\end{aligned} \tag{2·10}$$

Therefore, with the use of $\{\hat{D}^*_{\pm,0}, \hat{D}_{\pm,0}, \hat{I}_{\pm,0}, \hat{K}_{\pm,0}, \hat{M}_{\pm,0}\}$, it may also be possible to describe the $su(4)$ algebra.

The Casimir operator of the $su(4)$ algebra is defined as

$$\hat{\Gamma}_{su(4)} = \sum_{i=1}^{3}(\hat{S}^i\hat{S}_i + \hat{S}_i\hat{S}^i) + \sum_{i,j=1}^{3}\hat{S}^i_j\hat{S}^j_i - \left(\sum_{i=1}^{3}\hat{S}^i_i\right)^2/4. \tag{2·11}$$

The Casimir operators for the above three algebras are given as

$$\hat{\Gamma}_{so(5)} = 2(\hat{\boldsymbol{D}}^*\hat{\boldsymbol{D}} + \hat{\boldsymbol{D}}\hat{\boldsymbol{D}}^*) + \hat{\boldsymbol{I}}^2 + \hat{M}^2_0, \tag{2·12}$$

$$\hat{\Gamma}_{so(4)} = \hat{\boldsymbol{I}}^2 + \hat{\boldsymbol{K}}^2, \tag{2·13}$$

$$\hat{\Gamma}_{su(2)\otimes su(2)} = \hat{\boldsymbol{I}}^2 + \hat{\boldsymbol{M}}^2, \tag{2·14}$$

$$\hat{\boldsymbol{D}}^*\hat{\boldsymbol{D}} = \hat{D}^*_0\hat{D}_0 + (\hat{D}^*_+\hat{D}_+ + \hat{D}^*_-\hat{D}_-)/2,$$

$$\hat{\boldsymbol{D}}\hat{\boldsymbol{D}}^* = \hat{D}_0\hat{D}^*_0 + (\hat{D}_+\hat{D}^*_+ + \hat{D}_-\hat{D}^*_-)/2, \tag{2·15a}$$

$$\hat{\boldsymbol{I}}^2 = \hat{I}^2_0 + (\hat{I}_-\hat{I}_+ + \hat{I}_+\hat{I}_-)/2, \tag{2·15b}$$

$$\hat{\boldsymbol{K}}^2 = \hat{K}^2_0 + (\hat{K}_-\hat{K}_+ + \hat{K}_+\hat{K}_-)/2, \tag{2·15c}$$

$$\hat{\boldsymbol{M}}^2 = \hat{M}^2_0 + (\hat{M}_-\hat{M}_+ + \hat{M}_+\hat{M}_-)/2. \tag{2·15d}$$

The Casimir operator $\hat{\Gamma}_{su(4)}$ given in Eq. (2·11) can be rewritten as

$$\hat{\Gamma}_{su(4)} = 2(\hat{\boldsymbol{D}}^*\hat{\boldsymbol{D}} + \hat{\boldsymbol{D}}\hat{\boldsymbol{D}}^*) + \hat{\boldsymbol{I}}^2 + \hat{\boldsymbol{K}}^2 + \hat{\boldsymbol{M}}^2. \tag{2·16}$$

The $su(4)$ algebra corresponds to the case $M = 3$ presented in (I). We are interested in the $so(5)$, $so(4)$ and $su(2)\otimes su(2)$ algebras as sub-algebras of the $su(4)$ algebra. In order to obtain full irreducible representations of these three algebras, the

orthogonal sets must be specified by at least five, four and four quantum numbers, respectively. If we restrict ourselves to the case $N = 1$ ($p = 1$) considered in (I), the boson space is spanned by eight kinds of bosons $\hat{a}$, $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$, $\hat{b}$, $\hat{b}_1$, $\hat{b}_2$, $\hat{b}_3$. Here, the index p in the bosons $\hat{a}^p$, $\hat{b}_i^p$ used in (I) is omitted. Then, we can treat the orthogonal sets for the $so(5)$, $so(4)$ and $su(2) \otimes su(2)$ algebras as sub-algebras of the $su(4)$ algebra in the present boson space. However, the case $M = 3$, $N = 1$ cannot give a full irreducible representation for the $su(4)$ algebra, where at least nine quantum numbers are necessary. This case is represented by $M = 3$, $N = 2$, and it will be discussed in detail in a subsequent paper.

With the use of the boson representation of $\{\hat{E}^i, \hat{E}_i, \hat{E}_i^j, \hat{E}\}$ given by Eq. (I·2·2), $\{\hat{S}^i, \hat{S}_i, \hat{S}_i^j\}$ and $\hat{S}$ for the case $N = 1$ can be expressed as

$$\begin{aligned} \hat{S}^i &= \hat{a}_i^* \hat{b} + \hat{a}^* \hat{b}_i, \quad \hat{S}_i = \hat{b}^* \hat{a}_i + \hat{b}_i^* \hat{a}, \\ \hat{S}_i^j &= (\hat{a}_i^* \hat{a}_j - \hat{b}_j^* \hat{b}_i) + \delta_{ij}(\hat{a}^* \hat{a} - \hat{b}^* \hat{b}), \end{aligned} \tag{2·17}$$

$$\hat{S} = (\hat{b}^* \hat{b} - \hat{a}^* \hat{a}) - \sum_{i=1}^{3} (\hat{b}_i^* \hat{b}_i - \hat{a}_i^* \hat{a}_i). \tag{2·18}$$

In this paper, we treat the case $N = 1$. Therefore, as shown in (I), the $su(1,1)$ algebra can be introduced in our boson space. In (I), the generators of the $su(1,1)$ algebra are expressed in the form $\hat{T}^1$, $\hat{T}_1$, $\hat{T}_1^1$. In this paper, we adopt the notation $\hat{T}_{\pm,0}$, which represents the operators $\hat{T}_+ = \hat{T}^1$, $\hat{T}_- = \hat{T}_1$, $\hat{T}_0 = \hat{T}_1^1/2$. These operators can be expressed in terms of the Schwinger boson representation as follows:

$$\begin{aligned} \hat{T}_+ &= \hat{b}^* \hat{a}^* - \sum_{i=1}^{3} \hat{b}_i^* a_i^*, \quad \hat{T}_- = \hat{a}\hat{b} - \sum_{i=1}^{3} \hat{a}_i \hat{b}_i, \\ \hat{T}_0 &= (\hat{b}^* \hat{b} + \hat{a}^* \hat{a})/2 + \sum_{i=1}^{3} (\hat{b}_i^* \hat{b}_i + \hat{a}_i^* \hat{a}_i)/2 + 2. \end{aligned} \tag{2·19}$$

The Casimir operator is given by

$$\hat{\Gamma}_{su(1,1)} = \hat{T}_0^2 - (\hat{T}_- \hat{T}_+ + \hat{T}_+ \hat{T}_-)/2 \ (= \hat{\boldsymbol{T}}^2). \tag{2·20}$$

In (I), it was stressed that the generators of the $su(1,1)$ algebra commute with any of the $su(4)$ generators, $\hat{S}^i$, $\hat{S}_i$, $\hat{S}_i^j$. This fact is the most fundamental aspect of our treatment, and it plays an essential role in subsequent discussion. The Schwinger boson representation of $\{\hat{D}_{\pm,0}^*, \hat{D}_{\pm,0}, \hat{I}_{\pm,0}, \hat{K}_{\pm,0}, \hat{M}_{\pm,0}\}$ is given as

$$\begin{aligned} \hat{D}_+^* &= \hat{a}_3^* \hat{b} + \hat{a}^* \hat{b}_3, \quad \hat{D}_-^* = \hat{a}_2^* \hat{a}_1 - \hat{b}_1^* \hat{b}_2, \\ \hat{D}_0^* &= (\hat{a}_2^* \hat{b} + \hat{a}^* \hat{b}_2 - \hat{a}_3^* \hat{a}_1 + \hat{b}_1^* \hat{b}_3)/2, \\ \hat{D}_+ &= \hat{b}^* \hat{a}_3 + \hat{b}_3^* \hat{a}, \quad \hat{D}_- = \hat{a}_1^* \hat{a}_2 - \hat{b}_2^* \hat{b}_1, \\ \hat{D}_0 &= (\hat{b}^* \hat{a}_2 + \hat{b}_2^* \hat{a} - \hat{a}_1^* \hat{a}_3 + \hat{b}_3^* \hat{b}_1)/2, \\ \hat{I}_+ &= \hat{a}_1^* \hat{b} + \hat{a}^* \hat{b}_1 + \hat{a}_3^* \hat{a}_2 - \hat{b}_2^* \hat{b}_3, \\ \hat{I}_- &= \hat{b}^* \hat{a}_1 + \hat{b}_1^* \hat{a} + \hat{a}_2^* \hat{a}_3 - \hat{b}_3^* \hat{b}_2, \end{aligned} \tag{2·21}$$

$$\begin{aligned}
\hat{I}_0 &= (\hat{a}^*\hat{a} - \hat{b}^*\hat{b} + \hat{a}_1^*\hat{a}_1 - \hat{b}_1^*\hat{b}_1 \\
&\quad - \hat{a}_2^*\hat{a}_2 + \hat{b}_2^*\hat{b}_2 + \hat{a}_3^*\hat{a}_3 - \hat{b}_3^*\hat{b}_3)/2, && (2\cdot22)\\
\hat{K}_+ &= \hat{a}_1^*\hat{b} + \hat{a}^*\hat{b}_1 - \hat{a}_3^*\hat{a}_2 + \hat{b}_2^*\hat{b}_3, \\
\hat{K}_- &= \hat{b}^*\hat{a}_1 + \hat{b}_1^*\hat{a} - \hat{a}_2^*\hat{a}_3 + \hat{b}_3^*\hat{b}_2, \\
\hat{K}_0 &= (\hat{a}^*\hat{a} - \hat{b}^*\hat{b} + \hat{a}_1^*\hat{a}_1 - \hat{b}_1^*\hat{b}_1 \\
&\quad + \hat{a}_2^*\hat{a}_2 - \hat{b}_2^*\hat{b}_2 - \hat{a}_3^*\hat{a}_3 + \hat{b}_3^*\hat{b}_3)/2, && (2\cdot23)\\
\hat{M}_+ &= \hat{a}_2^*\hat{b} + \hat{a}^*\hat{b}_2 + \hat{a}_3^*\hat{a}_1 - \hat{b}_1^*\hat{b}_3, \\
\hat{M}_- &= \hat{b}^*\hat{a}_2 + \hat{b}_2^*\hat{a} + \hat{a}_1^*\hat{a}_3 - \hat{b}_3^*\hat{b}_1, && (2\cdot24)\\
\hat{M}_0 &= (\hat{a}^*\hat{a} - \hat{b}^*\hat{b} - \hat{a}_1^*\hat{a}_1 + \hat{b}_1^*\hat{b}_1 \\
&\quad + \hat{a}_2^*\hat{a}_2 - \hat{b}_2^*\hat{b}_2 + \hat{a}_3^*\hat{a}_3 - \hat{b}_3^*\hat{b}_3)/2. && (2\cdot24\text{a})
\end{aligned}$$

The above is our framework of the $so(5)$, $so(4)$ and $su(2)\otimes su(2)$ algebras as sub-algberas of the $su(4)$ algebra.

§3. The $so(5)$ algebra

Our primary interest concerns the investigation of the $so(5)$ algebra. In this section, we mainly consider this algebra. The generators in this case are given in the relations (2·21), (2·22) and (2·24a). For the present treatment, instead of the original notation it may be more transparent to adopt new notation for the bosons as follows:

$$\begin{aligned}
&\hat{a}\to\hat{a}_+,\quad \hat{b}\to\hat{b}_-,\quad \hat{a}_3\to\hat{a}_-,\quad \hat{b}_3\to\hat{b}_+,\\
&\hat{a}_1\to\hat{\beta}_-,\quad \hat{b}_1\to-\hat{\alpha}_+,\quad \hat{a}_2\to\hat{\alpha}_-,\quad \hat{b}_2\to\hat{\beta}_+. && (3\cdot1)
\end{aligned}$$

With the use of the above notation, we can set up the following relations for the operators $\hat{D}_\pm^*$, $\hat{D}_\pm$ and $\hat{T}_{\pm,0}$:

$$\hat{D}_+^* = \hat{s}_+,\quad \hat{D}_+ = \hat{s}_-,\quad \hat{D}_-^* = \hat{\sigma}_+,\quad \hat{D}_- = \hat{\sigma}_-. \qquad (3\cdot2)$$

$$\begin{aligned}
\hat{s}_+ &= \hat{a}_+^*\hat{b}_+ + \hat{a}_-^*\hat{b}_-,\quad \hat{s}_- = \hat{b}_+^*\hat{a}_+ + \hat{b}_-^*\hat{a}_-,\\
\hat{s}_0 &= (\hat{a}_+^*\hat{a}_+ - \hat{b}_+^*\hat{b}_+ + \hat{a}_-^*\hat{a}_- - \hat{b}_-^*\hat{b}_-)/2, && (3\cdot2\text{a})\\
\hat{\sigma}_+ &= \hat{\alpha}_+^*\hat{\beta}_+ + \hat{\alpha}_-^*\hat{\beta}_-,\quad \hat{\sigma}_- = \hat{\beta}_+^*\hat{\alpha}_+ + \hat{\beta}_-^*\hat{\alpha}_-,\\
\hat{\sigma}_0 &= (\hat{\alpha}_+^*\hat{\alpha}_+ - \hat{\beta}_+^*\hat{\beta}_+ + \hat{\alpha}_-^*\hat{\alpha}_- - \hat{\beta}_-^*\hat{\beta}_-)/2, && (3\cdot2\text{b})
\end{aligned}$$

$$\hat{T}_{\pm,0} = \hat{t}_{\pm,0} + \hat{\tau}_{\pm,0}, \qquad (3\cdot3)$$

$$\begin{aligned}
\hat{t}_+ &= \hat{a}_+^*\hat{b}_-^* - \hat{a}_-^*\hat{b}_+^*,\quad \hat{t}_- = \hat{b}_-\hat{a}_+ - \hat{b}_+\hat{a}_-,\\
\hat{t}_0 &= (\hat{a}_+^*\hat{a}_+ + \hat{b}_+^*\hat{b}_+ + \hat{a}_-^*\hat{a}_- + \hat{b}_-^*\hat{b}_-)/2 + 1, && (3\cdot3\text{a})\\
\hat{\tau}_+ &= \hat{\alpha}_+^*\hat{\beta}_-^* - \hat{\alpha}_-^*\hat{\beta}_+^*,\quad \hat{\tau}_- = \hat{\beta}_-\hat{\alpha}_+ - \hat{\beta}_+\hat{\alpha}_-,\\
\hat{\tau}_0 &= (\hat{\alpha}_+^*\hat{\alpha}_+ + \hat{\beta}_+^*\hat{\beta}_+ + \hat{\alpha}_-^*\hat{\alpha}_- + \hat{\beta}_-^*\hat{\beta}_-)/2 + 1. && (3\cdot3\text{b})
\end{aligned}$$

Further, we introduce the operators $\hat{R}_{\pm,0}$ in the form

$$\hat{R}_{\pm,0} = \hat{r}_{\pm,0} + \hat{\rho}_{\pm,0},\quad (\hat{R}_0 = \hat{S}/2) \qquad (3\cdot4)$$

$$\begin{aligned}
\hat{r}_+ &= -(\hat{a}_+^*\hat{a}_- + \hat{b}_+^*\hat{b}_-), \quad \hat{r}_- = -(\hat{a}_-^*\hat{a}_+ + \hat{b}_-^*\hat{b}_+), \\
\hat{r}_0 &= (\hat{a}_+^*\hat{a}_+ + \hat{b}_+^*\hat{b}_+ - \hat{a}_-^*\hat{a}_- - \hat{b}_-^*\hat{b}_-)/2, \qquad (3\cdot4\text{a}) \\
\hat{\rho}_+ &= \hat{\alpha}_+^*\hat{\alpha}_- + \hat{\beta}_+^*\hat{\beta}_-, \quad \hat{\rho}_- = \hat{\alpha}_-^*\hat{\alpha}_+ + \hat{\beta}_-^*\hat{\beta}_+, \\
\hat{\rho}_0 &= (\hat{\alpha}_+^*\hat{\alpha}_+ + \hat{\beta}_+^*\hat{\beta}_+ - \hat{\alpha}_-^*\hat{\alpha}_- - \hat{\beta}_-^*\hat{\beta}_-)/2. \qquad (3\cdot4\text{b})
\end{aligned}$$

Here, the operators $(\hat{r}_{\pm,0})$, $(\hat{s}_{\pm,0})$, $(\hat{t}_{\pm,0})$, $(\hat{\rho}_{\pm,0})$, $(\hat{\sigma}_{\pm,0})$ and $(\hat{\tau}_{\pm,0})$ were introduced in (II), and they obey the $su(2)$ algebras. The operators $\hat{D}_0^*$, $\hat{D}_0$, $\hat{I}_{\pm,0}$ and $\hat{M}_0$, which compose the $so(5)$ algebra, together with $\hat{D}_\pm^*$ and $\hat{D}_\pm$, are expressed as

$$\begin{aligned}
\hat{D}_0^* &= (\hat{a}_+^*\hat{\beta}_+ - \hat{a}_-^*\hat{\beta}_- - \hat{\alpha}_+^*\hat{b}_+ + \hat{\alpha}_-^*\hat{b}_-)/2, \\
\hat{D}_0 &= (\hat{\beta}_+^*\hat{a}_+ - \hat{\beta}_-^*\hat{a}_- - \hat{b}_+^*\hat{\alpha}_+ + \hat{b}_-^*\hat{\alpha}_-)/2, \qquad (3\cdot5) \\
\hat{I}_+ &= -\hat{a}_+^*\hat{\alpha}_+ + \hat{a}_-^*\hat{\alpha}_- - \hat{\beta}_+^*\hat{b}_+ + \hat{\beta}_-^*\hat{b}_-, \\
\hat{I}_- &= -\hat{\alpha}_+^*\hat{a}_+ + \hat{\alpha}_-^*\hat{a}_- - \hat{b}_+^*\hat{\beta}_+ + \hat{b}_-^*\hat{\beta}_-, \\
\hat{I}_0 &= (\hat{a}_+^*\hat{a}_+ - \hat{b}_+^*\hat{b}_+ + \hat{a}_-^*\hat{a}_- - \hat{b}_-^*\hat{b}_- \\
&\quad - \hat{\alpha}_+^*\hat{\alpha}_+ + \hat{\beta}_+^*\hat{\beta}_+ - \hat{\alpha}_-^*\hat{\alpha}_- + \hat{\beta}_-^*\hat{\beta}_-)/2, \qquad (3\cdot6) \\
\hat{M}_0 &= (\hat{a}_+^*\hat{a}_+ - \hat{b}_+^*\hat{b}_+ + \hat{a}_-^*\hat{a}_- - \hat{b}_-^*\hat{b}_- \\
&\quad + \hat{\alpha}_+^*\hat{\alpha}_+ - \hat{\beta}_+^*\hat{\beta}_+ + \hat{\alpha}_-^*\hat{\alpha}_- - \hat{\beta}_-^*\hat{\beta}_-)/2. \qquad (3\cdot7)
\end{aligned}$$

From the form (2·12), the Casimir operator $\hat{\Gamma}_{so(5)}$ can be rewritten as

$$\hat{\Gamma}_{so(5)} = \hat{\boldsymbol{T}}^2 + \hat{\boldsymbol{R}}^2 - 2, \qquad (3\cdot8)$$

$$\hat{\boldsymbol{R}}^2 = \hat{R}_0^2 + (\hat{R}_-\hat{R}_+ + \hat{R}_+\hat{R}_-)/2. \qquad (3\cdot9)$$

It should be again noted that the $so(5)$ generators $\hat{D}_{\pm,0}^*$, $\hat{D}_{\pm,0}$, $\hat{I}_{\pm,0}$, $\hat{M}_0$ commute not only with $\hat{T}_{\pm,0}$ but also with $\hat{R}_{\pm,0}$. This is an essential point for our treatment.

First, let us consider the following state:

$$\begin{aligned}
|k\kappa; TT_0, RR_0, m\mu\rangle &= (\hat{T}_+)^{T_0-T}(\hat{R}_+)^{R+R_0} \\
&\cdot (\hat{Q}^*)^{T-(k+\kappa+2)}(\hat{O}^*)^{k+\kappa-R} \\
&\cdot (\hat{D}_+^*)^{k+m}(\hat{D}_-^*)^{\kappa+\mu}(\hat{b}_-^*)^{2k}(\hat{\beta}_-)^{*2\kappa}|0\rangle. \qquad (3\cdot10)
\end{aligned}$$

The above form is copied from Eq. (II·22a). Here, the operators $\hat{s}_+$ and $\hat{\sigma}_+$ in Eq. (II·22a) are replaced by $\hat{D}_+^*$ and $\hat{D}_-^*$, respectively, and Q^* and O^* are defined in the form

$$\begin{aligned}
\hat{Q}^* &= +\hat{t}_+ \cdot \left(\sqrt{[\hat{t}_0^2 - (\hat{t}_-\hat{t}_+ + \hat{t}_+\hat{t}_-)/2] + 1/4} + 1/2 + \hat{t}_0\right)^{-1} \\
&\quad -\hat{\tau}_+ \cdot \left(\sqrt{[\hat{\tau}_0^2 - (\hat{\tau}_-\hat{\tau}_+ + \hat{\tau}_+\hat{\tau}_-)/2] + 1/4} + 1/2 + \hat{\tau}_0\right)^{-1}, \\
\hat{O}^* &= +\left(\sqrt{[\hat{r}_0^2 + (\hat{r}_-\hat{r}_+ + \hat{r}_+\hat{r}_-)/2] + 1/4} + 1/2 - \hat{r}_0\right)^{-1} \cdot \hat{r}_+ \\
&\quad -\left(\sqrt{[\hat{\rho}_0^2 + (\hat{\rho}_-\hat{\rho}_+ + \hat{\rho}_+\hat{\rho}_-)/2] + 1/4} + 1/2 - \hat{\rho}_0\right)^{-1} \cdot \hat{\rho}_+. \qquad (3\cdot10\text{a})
\end{aligned}$$

The discussion given in (II) seems to indicate that the states (3·10) compose an orthogonal set. It may be clear from $\hat{\Gamma}_{so(5)}$ given in Eq. (3·8) that the matrix elements of the $so(5)$ generators between two states $|k\kappa; TT_0, RR_0, m\mu\rangle$ and $|K'\kappa'; T'T'_0, R'R'_0, m'\mu'\rangle$ do not depend on T_0 and R_0 and vanish in the case that $T \neq T'$, $T_0 \neq T'_0$, $R \neq R'$, $R_0 \neq R'_0$.

A straightforward caculation leads to the relation

$$\begin{aligned}&(\hat{O}^*)^{k+\kappa-R}(\hat{b}^*_-)^{2k}(\hat{\beta}^*_-)^{2\kappa}|0\rangle \\ &\quad = (-)^{k+\kappa-R}(\hat{b}^*_+\hat{\beta}^*_- + \hat{b}^*_-\hat{\beta}^*_+)^{k+\kappa-R}\cdot(\hat{\beta}^*_-)^{R+(\kappa-k)}(\hat{b}^*_-)^{R-(\kappa-k)}|0\rangle. \quad (3\cdot11)\end{aligned}$$

Then, the state (3·10) can be rewritten as

$$|k\kappa; TT_0, RR_0, m\mu\rangle = (\hat{T}_+)^{T_0-T}(\hat{R}_+)^{R+R_0}|TR; k\kappa m\mu\rangle, \quad (3\cdot12)$$

$$|TR; k\kappa m\mu\rangle = (\hat{Q}^*)^{T-(k+\kappa+2)}(\hat{D}^*_+)^{k+m}(\hat{D}^*_-)^{\kappa+\mu}|k\kappa R\rangle, \quad (3\cdot13)$$

$$|k\kappa R\rangle = (-)^{k+\kappa-R}(\hat{b}^*_+\hat{\beta}^*_- + \hat{b}^*_-\hat{\beta}^*_+)^{k+\kappa-R}\cdot(\hat{\beta}^*_-)^{R+(\kappa-k)}(\hat{b}^*_-)^{R-(\kappa-k)}|0\rangle. \quad (3\cdot14)$$

In the case $T_0 = T$, $R_0 = -R$, the state (3·12) is reduced to the state (3·13). Then, if the treatment is restricted only to the $so(5)$ algebra, it may be sufficient to consider the state (3·13), which forms an orthogonal set. This is a possible orthogonal set for the $so(5)$ algebra. The operators $\hat{T}_+$, $\hat{R}_+$, $\hat{Q}^*$ and $\hat{D}^*_\pm$ are mutually commuted and satisfy

$$\hat{T}_-|k\kappa R\rangle = \hat{R}_-|k\kappa R\rangle = \hat{Q}|k\kappa R\rangle = \hat{D}_\pm|k\kappa R\rangle = 0. \quad (3\cdot15)$$

Therefore, the successive operation of $\hat{T}_+$, $\hat{R}_+$, $\hat{Q}^*$ and $\hat{D}^*_\pm$ on the state $|k\kappa R\rangle$ gives us the states (3·12) (and (3·13)).

We can prove that the state (3·10), i.e. (3·12), is an eigenstate of the operators $\hat{\boldsymbol{T}}^2$, $\hat{T}_0$, $\hat{\boldsymbol{R}}^2$, $\hat{R}_0$, $\hat{I}_0$ and $\hat{M}_0$ with the eigenvalues $T(T-1)$, T_0, $R(R+1)$, R_0, $(m-\mu)$ and $(m+\mu)$, respectively. Therefore, the eigenvalue of $\hat{\Gamma}_{so(5)}$ is given by $T(T-1) + R(R+1) - 2[= (T-2)(T+1) + R(R+1)]$. Denoting the eigenvalues of $\hat{I}_0$ and $\hat{M}_0$ as I_0 and M_0, m and μ are given by $m = (M_0 + I_0)/2$ and $\mu = (M_0 - I_0)/2$, respectively. Hereafter, we denote $|k\kappa; TT_0, RR_0, m\mu\rangle$ as

$$|k\kappa; TT_0, RR_0, m\mu\rangle = |k\kappa; TT_0, RR_0, I_0, M_0\rangle\rangle, \quad (3\cdot16)$$

$$m = (M_0 + I_0)/2, \quad \mu = (M_0 - I_0)/2. \quad (3\cdot17)$$

The state (3·16) is not an eigenstate of the operator $\hat{\boldsymbol{I}}^2$. Then, it may be interesting to investigate the orthogonal set consisting of the eigenstates of $\hat{\boldsymbol{I}}^2$. An eigenstate of $\hat{\boldsymbol{I}}^2$ may be expressed in terms of possible linear combinations of k and κ with respect to the states (3·16), and the method developed in Ref. 5) helps us to construct the eigenstates. First, we note the following relation:

$$\begin{aligned}|J; TT_0, RR_0, II_0, M_0\rangle\rangle &= (\hat{T}_+)^{T_0-T}(\hat{R}_+)^{R+R_0}(\hat{\boldsymbol{D}}^{*2})^{(T+M_0-J-2)/2} \\ &\quad\cdot\sum_{J_0R'_0}\langle JJ_0RR'_0|II_0\rangle Z_{JJ_0}(\hat{\boldsymbol{D}}^*)|TRR'_0\rangle\rangle, \quad (3\cdot18)\end{aligned}$$

$$|TRR'_0\rangle\rangle = (\hat{\beta}^*_-)^{R+R'_0}(\hat{b}^*_-)^{R-R'_0}\cdot(\hat{b}^*_+\hat{\beta}^*_- + \hat{b}^*_-\hat{\beta}^*_+)^{T-R-2}|0\rangle. \quad (3\cdot19)$$

The operator $\hat{\boldsymbol{D}}^{*2}$ is defined as

$$\hat{\boldsymbol{D}}^{*2} = \hat{D}^{*2}_0 + (\hat{D}^*_+\hat{D}^*_- + \hat{D}^*_-\hat{D}^*_+)/2. \quad (3\cdot20)$$

The solid harmonic $Z_{JJ_0}(r\theta\phi) = r^J Y_{JJ_0}(\theta\phi)$ is a homogeneous polynomial for $r_{\pm,0}$ with $r_\pm = x \pm iy = r\sin\theta\exp(\pm i\phi)$ and $r_0 = z = r\cos\theta$. By replacing $r_{\pm,0}$ in the solid harmonics with $\hat{D}^*_{\pm,0}$, we obtain $Z_{JJ_0}(\hat{\boldsymbol{D}}^*)$. It may be clear that $\hat{\boldsymbol{D}}^{*2}$ and $Z_{JJ_0}(\hat{\boldsymbol{D}}^*)$ are a scalar and a tensor of rank J for the $su(2)$-generators $\hat{I}_{\pm,0}$, respectively. The state $|TRR'_0\rangle\rangle$ is an eigenstate of $\hat{\boldsymbol{I}}^2$ and $\hat{I}_0$ with the eigenvalues $R(R+1)$ and R'_0, respectively. The quantity $\langle JJ_0RR'_0|II_0\rangle$ denotes the Clebsch-Gordan coefficient. From the above argument, we can understand that the state (3·18) is an eigenstate of the operator $\hat{\boldsymbol{I}}^2$ with eigenvalue $I(I+1)$. Of course, it is also an eigenstate of $\hat{\boldsymbol{T}}^2$, T_0, $\hat{\boldsymbol{R}}^2$, $\hat{R}_0$, $\hat{I}_0$ and $\hat{M}_0$ with eigenvalues $T(T-1)$, T_0, $R(R+1)$, R_0, I_0 and M_0. However, it should be noted that although the states (3·18) form a linearly independent set, they do not, in general, compose an orthogonal set. Then, under an appropriate linear combination of the states (3·18) for J, for example, using the Schmidt method, we obtain an orthogonal set in the following form:

$$|\lambda; TT_0, RR_0, II_0, M_0\rangle\rangle = \sum_J C_{\lambda J}|J; TT_0, RR_0, II_0, M_0\rangle\rangle. \tag{3·21}$$

Here, the quantity λ denotes a quantum number which supports the orthogonality of the states (3·21). The coefficient $C_{\lambda J}$ may depend in T, R, I and M_0. Under the conditions $J \le T + M_0 - 2$ and $|I - R| \le J \le I + R$, the summation over J is performed. Then, the state (3·21) can be rewritten in the form

$$|\lambda; TT_0, RR_0, II_0, M_0\rangle\rangle = (\hat{T}_+)^{T_0-T}(\hat{R}_+)^{R+R_0}|\lambda TR; II_0, M_0\rangle\rangle, \tag{3·22}$$

$$|\lambda TR; II_0, M_0\rangle\rangle = \sum_J (\hat{\boldsymbol{D}}^{*2})^{(T+M_0-J-2)/2} \cdot \sum_{J_0R'_0} \langle JJ_0RR'_0|II_0\rangle Z_{JJ_0}(\boldsymbol{D}^{*2})|TRR'_0\rangle\rangle. \tag{3·23}$$

If restricted to the $so(5)$ algebra, it may be sufficient to consider the state (3·23), which corresponds to the case $T_0 = T$, $R_0 = -R$ in the state (3·22). The state $|TRR'_0\rangle\rangle$ obeys the condition

$$\hat{T}_-|TRR'_0\rangle\rangle = \hat{R}_-|TRR'_0\rangle\rangle = \hat{D}_{\pm,0}|TRR'_0\rangle\rangle = 0. \tag{3·24}$$

The relation (3·24) should be compared with the relation (3·15). In the state $|TRR'_0\rangle\rangle$, not only the condition $\hat{D}_\pm|TRR'_0\rangle\rangle = 0$ but also the condition $\hat{D}_0|TRR'_0\rangle\rangle = 0$ is taken into account. The appropriate superposition of states obtained by the successive operation of $\hat{D}^*_{\pm,0}$ ($\hat{T}_+$ and $\hat{R}_+$) on the state $|TRR'_0\rangle\rangle$ gives us the state (3·23) [(3·22)]. It is interesting to note that in the state (3·23) [(3·22)], not only $\hat{D}^*_\pm$ but also $\hat{D}^*_0$ is taken into account symmetrically. Further, we have the relations

$$\hat{\boldsymbol{I}}^2 = |TRR'_0\rangle\rangle = R(R+1)|TRR'_0\rangle\rangle, \tag{3·25}$$

$$\hat{\boldsymbol{R}}^2|\lambda TR; II_0, M_0\rangle\rangle = R(R+1)|\lambda TR; II_0, M_0\rangle\rangle. \tag{3·26}$$

Therefore, $\hat{R}_{\pm,0}$ play the role of the operators for the reduced iso-spin introduced in the charge-independent pairing theory. Of course, $\hat{R}_0$ is identical to $-\hat{S}/2$, which commutes with all the $su(4)$ generators.

§4. The $so(4)$ and $su(2)\otimes su(2)$ algebras

It is convenient to discuss the two algebras $so(4)$ and $su(2)\otimes su(2)$ on the same footing. Let us start with the $so(4)$ algebra. In the present case, the generators are listed in the relations (2·22) and (2·23). As in the previous case, we adopt the following new notation for bosons:

$$\begin{aligned}&\hat{a}\to\hat{a}_+,\quad \hat{b}\to\hat{b}_-,\quad \hat{a}_1\to\hat{a}_-,\quad \hat{b}_1\to\hat{b}_+,\\ &\hat{a}_2\to\hat{\beta}_-,\quad \hat{b}_2\to-\hat{\alpha}_+,\quad \hat{a}_3\to\hat{\alpha}_-,\quad \hat{b}_3\to\hat{\beta}_+.\end{aligned}\tag{4·1}$$

With the use of the above notation, we can set up the following relation for the $so(4)$ generators $\hat{I}_{\pm,0}$ and $\hat{K}_{\pm,0}$:

$$\hat{I}_{\pm,0}=\hat{J}_{\pm,0}+\hat{L}_{\pm,0}\ (=\hat{S}_{\pm,0}),\tag{4·2}$$

$$\hat{K}_{\pm,0}=\hat{J}_{\pm,0}-\hat{L}_{\pm,0},\tag{4·3}$$

$$\hat{J}_{\pm,0}=\hat{s}_{\pm,0},\quad \hat{L}_{\pm,0}=\hat{\sigma}_{\pm,0}.\tag{4·4}$$

The sets $\{\hat{J}_{\pm,0}\}$ and $\{\hat{L}_{\pm,0}\}$ obey the commutation relations of the $su(2)$ algebra and this shows that the $so(4)$ algebra can be treated as the $su(2)\otimes su(2)$ algebra. This is a well-known fact. The Casimir operator $\hat{\Gamma}so(4)$ given in Eq. (2·13) can be rewritten as

$$\hat{\Gamma}_{so(4)}=2(\hat{\boldsymbol{J}}^2+\hat{\boldsymbol{L}}^2),\tag{4·5}$$

$$\begin{aligned}\hat{\boldsymbol{J}}^2&=\hat{J}_0^2+(\hat{J}_-\hat{J}_++\hat{J}_+\hat{J}_-)/2,\\ \hat{\boldsymbol{L}}^2&=\hat{L}_0^2+(\hat{L}_-\hat{L}_++\hat{L}_+\hat{L}_-)/2.\end{aligned}\tag{4·6}$$

The operators $\hat{s}_{\pm,0}$ and $\hat{\sigma}_{\pm,0}$ are of the same forms as those given in the relations (3·2a) and (3·2b). The $su(1,1)$ generators $\hat{T}_{\pm,0}$ and the $su(2)$ generators $\hat{R}_{\pm,0}$ are also given as

$$\hat{T}_{\pm,0}=\hat{t}_{\pm,0}+\hat{\tau}_{\pm,0},\tag{4·7}$$

$$\hat{R}_{\pm,0}=\hat{r}_{\pm,0}+\hat{\rho}_{\pm,0}.\tag{4·8}$$

The operators $\hat{t}_{\pm,0}$, $\hat{\tau}_{\pm,0}$, $\hat{r}_{\pm,0}$ and $\hat{\rho}_{\pm,0}$ are of the same forms as those given in the relations (3·3a), (3·3b), (3·4a) and (3·4b), repectively.

Then, by copying from Eq. (II·22a), we set up the following state:

$$\begin{aligned}|TT_0,RR_0;JJ_0,LL_0\rangle\rangle&=(\hat{T}_+)^{T_0-T}(\hat{R}_+)^{R+R_0}\\ &\quad\cdot(\hat{Q}^*)^{T-k-\kappa-2}(\hat{O}^*)^{k+\kappa-R}\cdot|JJ_0,LL_0\rangle\rangle,\end{aligned}\tag{4·9}$$

$$|JJ_0,LL_0\rangle\rangle=(\hat{J}_+)^{k+m}(\hat{L}_+)^{\kappa+\mu}(\hat{b}_-^*)^{2k}(\hat{\beta}_-^*)^{2\kappa}|0\rangle.\tag{4·10}$$

The state (4·9) is of the same form as that given in the relation (3·10) under the replacements $\hat{D}_+^*\to\hat{J}_+$, $\hat{D}_-^*\to\hat{L}_+$, $k\to J$, $m\to J_0$, $\kappa\to L$, $\mu\to L_0$. The state (4·9) is an eigenstate of $\hat{\boldsymbol{T}}^2$, $\hat{T}_0$, $\hat{\boldsymbol{R}}^2$, $\hat{R}_0$, $\hat{\boldsymbol{J}}^2$, $\hat{J}_0$, $\hat{\boldsymbol{L}}^2$ and $\hat{L}_0$ with an eigenvalues $T(T-1)$, T_0, $R(R+1)$, R_0, $J(J+1)$, J_0, $L(L+1)$ and L_0, respectively. Then, the eigenvalue of the Casimir operator $\Gamma_{so(4)}$ is given as $2[J(J+1)+L(L+1)]$.

It is interesting to construct the eigenstate of $\hat{\boldsymbol{I}}^2$ and $\hat{I}_0$. With the use of the form (II·19), we can construct the eigenstate with the eigenvalues $I(I+1)$ and I_0, respectively, from the form (4·10):

$$\begin{aligned}|TT_0, RR_0, JL, II_0\rangle\rangle &= (\hat{T}_+)^{T_0-T}(\hat{R}_+)^{R+R_0}(\hat{I}_+)^{I+I_0} \\ &\quad \cdot (\hat{Q}^*)^{T-J-L-2}(\hat{O}^*)^{J+L-R}(\hat{P}^*)^{J+L-I} \cdot (\hat{b}^*_-)^{2J}(\hat{\beta}^*_-)^{2L}|0\rangle \\ &= (\hat{T}_+)^{T_0-T}(\hat{R}_+)^{R+R_0}(\hat{Q}^*)^{T-J-L-2}(\hat{O}^*)^{J+L-R} \cdot |JL, II_0\rangle\rangle,\end{aligned} \tag{4·11}$$

$$\hat{P}^* = +\left(\sqrt{\hat{\boldsymbol{J}}^2 + 1/4} + 1/2 - \hat{J}_0\right)^{-1} \cdot J_+ - \left(\sqrt{\hat{\boldsymbol{L}}^2 + 1/4} + 1/2 - \hat{L}_0\right)^{-1} \cdot L_+, \tag{4·11a}$$

$$|JL, II_0\rangle\rangle = (\hat{I}_+)^{I+I_0}|JL, I\rangle\rangle, \tag{4·12}$$

$$|JL, I\rangle\rangle = (\hat{a}^*_-\hat{\beta}^*_- - \hat{b}^*_-\hat{\alpha}^*_-)^{J+L-I} \cdot (\hat{\beta}^*_-)^{I+(L-J)}(\hat{b}^*_-)^{I-(L-J)}|0\rangle. \tag{4·13}$$

Here, $\hat{s}_{\pm,0}$, k and κ in (II) correspond to $\hat{I}_{\pm,0}$, J and L in the present notation, respectively. The operators $\hat{Q}^*$ and $\hat{O}^*$ in the relation (4·11) are given in Eq. (3·10a) and $\hat{J}^2$ and $\hat{L}^2$ are defined in Eq. (4·6). In the case of the $so(4)$ algebra, only the state (4·12) is necessary. It is obtained from the state (4·11) by setting $T_0 = T$ and $R_0 = -R$. We can prove that the state (4·12) satisfies

$$\hat{\boldsymbol{K}}^2|JL, II_0\rangle\rangle = [2(J(J+1) + L(L+1)) - I(I+1)]|JL, II_0\rangle\rangle. \tag{4·14}$$

Of course, the state (4·11) satisfies the same relation as that given in Eq. (4·14).

Next, we discuss the $su(2) \otimes su(2)$ algebra. The generators are I listed in the relations (2·22) and (2·24). As the previous two cases, we adopt the following notation for bosons:

$$\begin{aligned}&\hat{a}_1 \to \hat{b}_+, \quad \hat{a}_2 \to -\hat{a}_-, \quad \hat{a}_3 \to \hat{a}_+, \quad \hat{b} \to -b_-, \\ &\hat{a} \to \alpha_+, \quad \hat{b}_1 \to \hat{\alpha}_-, \quad \hat{b}_2 \to \hat{\beta}_+, \quad \hat{b}_3 \to -\hat{\beta}_-.\end{aligned} \tag{4·15}$$

In the present notation, we can set up the following relations for the generators $\hat{I}_{\pm,0}$ and $\hat{M}_{\pm,0}$:

$$\hat{I}_{\pm,0} = \hat{r}_{\pm,0} + \hat{\rho}_{\pm,0} \ (= \hat{R}_{\pm,0}), \tag{4·16}$$

$$\hat{M}_{\pm,0} = \hat{s}_{\pm,0} + \hat{\sigma}_{\pm,0} \ (= \hat{S}_{\pm,0}). \tag{4·17}$$

Here, $\hat{s}_{\pm,0}$, $\hat{\sigma}_{\pm,0}$, $\hat{r}_{\pm,0}$ and $\hat{\rho}_{\pm,0}$ are of the same form as those given in the relations (3·2a), (3·2b), (3·4a) and (3·4b). In associating with the above operators, we can introduce the operators $\hat{t}_{\pm,0}$ and $\hat{\tau}_{\pm,0}$ in the same forms as those given in the relations (3·3a) and (3·3b). Expressed in the original notation, these operators are written:

$$\begin{aligned}\hat{t}_+ &= -\hat{a}^*_3\hat{b}^* + \hat{a}^*_1\hat{a}^*_2, \quad t_- = -\hat{b}\hat{a}_3 + \hat{a}_2\hat{a}_1, \\ \hat{t}_0 &= (\hat{a}^*_3\hat{a}_3 + \hat{a}^*_2\hat{a}_2 + \hat{a}^*_1\hat{a}_1 + \hat{b}^*\hat{b})/2 + 1,\end{aligned} \tag{4·18a}$$

$$\begin{aligned}\hat{\tau}_+ &= -\hat{a}^*\hat{b}^*_3 - \hat{b}^*_2\hat{b}^*_1, \quad \hat{\tau}_- = -ba - b_1b_2, \\ \hat{\tau}_0 &= (\hat{a}^*\hat{a} + \hat{b}^*_1\hat{b}_1 + \hat{b}^*_2\hat{b}_2 + \hat{b}^*_3\hat{b}_3) + 1.\end{aligned} \tag{4·18b}$$

For the above operators, the following relation should be noted:

$$\hat{t}_\pm + \hat{\tau}_\pm \neq \hat{T}_\pm, \quad \hat{t}_0 + \hat{\tau}_0 = \hat{T}_0. \tag{4·19}$$

Applying the relation (II·21b) to the present case, we have the state

$$\begin{aligned}|k\kappa; II_0, MM_0, n\nu\rangle &= (\hat{t}_+)^{n-k-1}(\hat{\tau}_+)^{\nu-\kappa-1} \\ &\cdot (\hat{I}_+)^{I+I_0}(\hat{M}_+)^{M+M_0}(\hat{P}^*)^{k+\kappa-M}(\hat{O}^*)^{k+\kappa-I} \cdot (\hat{b}^*_-)^{2k}(\hat{\beta}^*_-)^{2\kappa}|0\rangle.\end{aligned} \tag{4·20}$$

The state (4·20) is an eigenstate of $\hat{\boldsymbol{I}}^2$, $\hat{I}_0$, $\hat{\boldsymbol{M}}^2$ and $\hat{M}_0$ with eigenvalues $I(I+I)$, I_0, $M(M+1)$ and M_0, respectively. However, the relation (4·19) shows that with direct use of the method in (II), we are not able to construct the eigenstate of $\hat{\boldsymbol{T}}^2$ and $\hat{T}_0$. Certainly, $\hat{T}_{\pm,0}$ can be expressed as

$$\begin{aligned}\hat{T}_+ &= \hat{a}^*_+\hat{\beta}^*_- + \hat{a}^*_-\hat{\beta}^*_+ - \hat{b}^*_+\hat{\alpha}^*_- - \hat{b}^*_-\hat{\alpha}^*_+, \\ \hat{T}_- &= \hat{\beta}_-\hat{a}_+ + \hat{\beta}_+\hat{a}_- - \hat{\alpha}_-\hat{b}_+ - \hat{\alpha}_+\hat{b}_-, \\ \hat{T}_0 &= (\hat{a}^*_+\hat{a}_+ + \hat{a}^*_-\hat{a}_- + \hat{b}^*_+\hat{b}_+ + \hat{b}^*_-\hat{b}_- \\ &\quad + \hat{\alpha}^*_+\hat{\alpha}_+ + \hat{\alpha}^*_-\hat{\alpha}_- + \hat{\beta}^*_+\hat{\beta}_+ + \hat{\beta}^*_-\hat{\beta}_-)/2 + 2.\end{aligned} \tag{4·21}$$

The state (4·20) is an eigenstate of $\hat{T}_0$ with eigenvalue $T_0 = n + \nu$. Further, it is also an eigenstate of $\hat{\Lambda}_0$ ($= \hat{S}/2$) with the eigenvalue $\Lambda_0 = n - \nu$. Here, $\hat{\Lambda}_0$ is defined by

$$\begin{aligned}\hat{\Lambda}_0 = (&\hat{a}^*_+\hat{a}_+ + \hat{a}^*_-\hat{a}_- + \hat{b}^*_+\hat{b}_+ + \hat{b}^*_-\hat{b}_- \\ &- \hat{\alpha}^*_+\hat{\alpha}_+ - \hat{\alpha}^*_-\hat{\alpha}_- - \hat{\beta}^*_+\hat{\beta}_+ - \hat{\beta}^*_-\hat{\beta}_-)/2.\end{aligned} \tag{4·22}$$

Then, n and ν are given as

$$n = (T_0 + \Lambda_0)/2, \quad \nu = (T_0 - \Lambda_0)/2. \tag{4·23}$$

With the above conditions, the following state is introduced:

$$\begin{aligned}|\lambda T; IM\Lambda_0\rangle\rangle = \sum_{k\kappa} &C_{\lambda T,k\kappa}(\hat{t}_+)^{(T+\Lambda_0)/2-k-1}(\hat{\tau}_+)^{(T-\Lambda_0)/2-\kappa-1} \\ &\cdot (\hat{P}^*)^{k+\kappa-M}(\hat{O}^*)^{k+\kappa-I}(\hat{b}^*_-)^{2k}(\hat{\beta}^*_-)^{2\kappa}|0\rangle.\end{aligned} \tag{4·24}$$

Independently of the coefficient $C_{\lambda T,k\kappa}$, the state (4·24) satisfies

$$\hat{T}_0|\lambda T; IM\Lambda_0\rangle\rangle = T|\lambda T; IM\Lambda_0\rangle\rangle. \tag{4·25}$$

Then, we require that the coefficient $C_{\lambda T,k\kappa}$ be determined by the condition

$$\hat{T}_-|\lambda T; IM\Lambda_0\rangle\rangle = 0. \tag{4·26}$$

Thus, the following state is sufficient for our purposes

$$|\lambda\Lambda_0; TT_0, II_0, MM_0\rangle\rangle = (\hat{T}_+)^{T_0-T}|\lambda\Lambda_0; II_0, MM_0\rangle\rangle, \tag{4·27}$$

$$|\lambda\Lambda_0; II_0, MM_0\rangle\rangle = (\hat{I}_+)^{I+I_0}(\hat{M}_+)^{M+M_0}|\lambda T; IM\Lambda_0\rangle\rangle. \tag{4·28}$$

The eigenvalue of the Casimir operator $\hat{\Gamma}_{su(2)\otimes su(2)}$ is $I(I+1)+M(M+1)$. The coefficient $C_{\lambda T,k\kappa}$ may also depend on the quantum numbers Λ_0, I and M. In the manner discussed above, we can construct an orthogonal set for the $su(2)\otimes su(2)$ algebra. If we restrict ourselves to the $su(2)\otimes su(2)$ algebra, it is sufficient to consider the state (4·28) obtained from the state (4·27) for $T_0=T$. Clearly, the state (4·28) is an eigenstate of the operators $\hat{\boldsymbol{I}}^2$, $\hat{I}_0$, $\hat{\boldsymbol{M}}^2$ and $\hat{M}_0$ with eigenvalues $I(I+1)$, I_0, $M(M+1)$ and M_0, respectively. It is interesting to compare the state (4·28) with the state (4·10). Both are deeply related to the $su(2)\otimes su(2)$ algebra. The orthogonal set for the previous case is given by $|JJ_0,LL_0\rangle\rangle$. On the other hand, the latter is given by $|\lambda\Lambda_0;II_0,MM_0\rangle\rangle$, which depends on the two quantum numbers λ and Λ_0. This implies that it may be impossible to separate the part related to λ and Λ_0 from the part related to II_0 and MM_0. However, in the previous case, it is possible.

§5. Discussion

In §§3 and 4, we presented a possible boson realization for the $so(5)$, $so(4)$ and $su(2)\otimes su(2)$ algebras as three sub-algebras of the $su(4)$ algebra. As mentioned in §1, our motivation is to give a possible boson realization of the nucleon pair system given in §1. Therefore, it may be necessary to investigate the correspondence between the fermion and boson systems. For this purpose, we investigate the vacuum.

Let us start with the $so(5)$ algebra. In the fermion system considered in §1, the vacuum $|\phi_0)$, which satisfied $\hat{n}_m|\phi_0)=\hat{p}_m|\phi_0)=0$, obeys the following conditions in the present case:

$$\dot{D}_{\pm,0}|\phi_0) = \dot{I}_{\pm,0}|\phi_0)=0, \tag{5·1a}$$
$$\dot{M}_0|\phi_0) = -\Omega|\phi_0), \tag{5·1b}$$
$$\dot{\Gamma}_{so(5)}|\phi_0) = \Omega(\Omega+3)|\phi_0). \tag{5·1c}$$

We consider the state (3·19) for the case $R=R_0'=0$, which we denote as $|\phi_1\rangle\rangle$:

$$|\phi_1\rangle\rangle = |T,R=0,R_0'=0\rangle\rangle = (\hat{b}_+^*\hat{\beta}_-^*+\hat{b}_-^*\hat{\beta}_+^*)^{T-2}|0\rangle. \tag{5·2}$$

It is easily proved that the state $|\phi_1\rangle\rangle$ obeys

$$\hat{D}_{\pm,0}|\phi_1\rangle\rangle = \hat{I}_{\pm,0}|\phi_1\rangle\rangle=0, \tag{5·3a}$$
$$\hat{M}_0|\phi_1\rangle\rangle = -(T-2)|\phi_1\rangle\rangle, \tag{5·3b}$$
$$\hat{\Gamma}_{so(5)} = (T-2)(T+1)|\phi_1\rangle\rangle. \tag{5·3c}$$

Then, the state $|\phi_1\rangle\rangle$ with the condition $T-2=\Omega$ corresponds to the vacuum $|\phi_0)$. This implies that our boson representation of the $so(5)$ algebra is simply a possible boson realization of the $so(5)$ algebra expressed by the nucleon pair operators. In the case of the $so(4)$ algebra, the vacuum $|\phi_0)$ obeys the conditions

$$\dot{I}_{\pm,0}|\phi_0)=\dot{K}_{\pm,0}|\phi_0)=0, \tag{5·4a}$$
$$\dot{\Gamma}_{so(4)}|\phi_0=0. \tag{5·4b}$$

We consider the state (4·13) for the case $J = L = I = 0$, which we denote as $|\phi_2\rangle\rangle$:

$$|\phi_2\rangle\rangle = |J = 0, L = 0, I = 0\rangle\rangle = |0\rangle. \tag{5·5}$$

The state $|\phi_2\rangle\rangle$ obeys the same conditions as those given in the relation (5·4). Therefore, we conclude that the state $|\phi_2\rangle\rangle$ $(= |0\rangle)$ corresponds to $|\phi_0)$. This implies that the present boson representation corresponds to the nucleon system obeying the $so(4)$ algebra. Finally, we investigate the case of the $su(2) \otimes su(2)$ algebra. In this case, the vacuum $|\phi_0)$ obeys

$$\dot{I}_{\pm,0}|\phi_0) = \dot{M}_-|\phi_0) = 0, \tag{5·6a}$$
$$\dot{M}_0|\phi_0) = -\Omega|\phi_0), \tag{5·6b}$$
$$\dot{\Gamma}_{su(2)\otimes su(2)}|\phi_0) = \Omega(\Omega+1)|\phi_0). \tag{5·6c}$$

In order to obtain the state $|\phi_3\rangle\rangle$ that corresponds to $|\phi_0)$, let us consider the state (4·24) for the case $I = 0$, $M = T - 2$, $\Lambda_0 = 0$. As a possible form satisfying the conditions (4·25) and (4·26), we can consider the following form (which we denote as $|\phi_3\rangle\rangle$):

$$\begin{aligned} |\phi_3\rangle\rangle &= |\lambda T; I = 0, M = T - 2, \Lambda_0 = 0\rangle\rangle \\ &= (\hat{b}^*_+\hat{\beta}^*_- + \hat{b}^*_-\hat{\beta}^*_+)^{T-2}|0\rangle. \end{aligned} \tag{5·7}$$

If $T - 2 = \Omega$, the state $|\phi_3\rangle\rangle$ obeys the same conditions as those given in the relations (5·6), and it is concluded that the present boson representation is equivalent to the nucleon system obeying the $su(2) \otimes su(2)$ algebra.

In the above, we obtained the three states $|\phi_1\rangle\rangle$, $|\phi_2\rangle\rangle$ and $|\phi_3\rangle\rangle$, which correspond to the nucleon vacuum $|\phi_0)$ for the $so(5)$, $so(4)$ and $su(2) \otimes su(2)$ algebras, respectively. The state $|\phi_3\rangle\rangle$ is identical to $|\phi_1\rangle\rangle$, but it is distinct from the state $|\phi_2\rangle\rangle$. Therefore, in the framework of each algebra, $|\phi_1\rangle\rangle$, $|\phi_2\rangle\rangle$ and $|\phi_3\rangle\rangle$ play the same role as that of $|\phi_0)$. This implies that for separate treatment of the three algebras, it may be sufficient to consider the three states $|\phi_1\rangle\rangle$, $|\phi_2\rangle\rangle$ and $|\phi_3\rangle\rangle$ for the $so(5)$, $so(4)$ and $su(2) \otimes su(2)$ algebras, respectively. For the treatment of the $su(4)$ algebra in the fermion space, $|\phi_0)$ plays the role of the vacuum, and it obeys the conditions

$$\dot{D}_{\pm,0}|\phi_0) = \dot{I}_{\pm,0}|\phi_0) = \dot{K}_{\pm,0}|\phi_0) = \dot{M}_-|\phi_0) = 0, \tag{5·8a}$$
$$\dot{M}_0|\phi_0) = -\Omega|\phi_0), \tag{5·8b}$$
$$\dot{\Gamma}_{su(4)}|\phi_0) = \Omega(\Omega+4)|\phi_0). \tag{5·8c}$$

However, from the above discussion we can conclude that in the frame adopted in this paper, it may be impossible to treat the Schwinger-type boson realization of the $su(4)$ algebra in relation to the nucleon pair system obeying the $su(4)$ algebra, because there does not exist a state such as $|\phi_0)$ that simultaneously satisfies the conditions (5·8). In a subsequent paper, we will discuss a possible form of the Schwinger-type boson realization of the $su(4)$ algebra, which enables us to connect with the nucleon pair system. We now sketch the basic idea. In the present paper, the case $M = 3$,

$N = 1$ was treated. Then, in the frame of $M = 3$, $N = 2$, we define the state

$$|\phi_0\rangle\rangle = (\hat{b}_2^{1*}\hat{b}_3^{2*} - \hat{b}_3^{1*}\hat{b}_2^{2*})^{\Omega}|0\rangle. \tag{5·9}$$

It is possible to prove that the state $|\phi_0\rangle\rangle$ satisfies the relations given in Eqs. (5·8). For the reason given above, in a subsequent paper, we will present the treatment of the $su(4)$ algebra to the case $M = 3$, $N = 2$.

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References

1) S. T. Belyaev and V. G. Zelevinsky, Nucl. Phys. **39** (1962), 582.
2) T. Marumori, M. Yamamura and A. Tokunaga, Prog. Theor. Phys. **31** (1964), 1009.
3) A. Klein and E. R. Marshalek, Rev. Mod. Phys. **63** (1991), 375.
4) Special issue: International Conference on the Physics of Nuclear Structure at the Extremes, Lewes, Sussex, UK. 17-19, June 1998, J. of Phys. G, Nucl. Part. Phys. **25** (1998), 577.
5) J. Eichler and M. Yamamura, Nucl. Phys. **A182** (1972), 33.
6) S. -C. Zhang, Science **275** (1997), 1089.
7) M. Guidry, L. -A. Wu, C. -L. Wu and Y. Sun, preprint, cond-mat/9903150.
L. -A. Wu, M. Guidry, Y. Sun and C. -L. Wu, preprint, cond-mat/9905436.
Y. Sun, M. Guidry, L. -A. Wu and C. -L. Wu, preprint, cond-mat/9912072.
8) A. Kuriyama, J. da Providência and M. Yamamura, Prog. Theor. Phys. **103** (2000), 285.
9) A. Kuriyama, J da Providência and M. Yamamura, Prog. Theor. Phys. **104** (2000), 259.