

## Divergence, Spacetime Dimension and Fractal Structure

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With a Cantor spacetime in mind, we assume the dimension of spacetime to be slightly smaller than four. Within the framework of QED, this dimension can be determined by calculating Feynman diagrams. We infer that the dimension of spacetime may be influenced by holes in space.

The dimension of spacetime<sup>1)</sup> is one of the fundamental concepts in physics and should be determined<sup>2)</sup> from a truly fundamental theory of everything.

We take for granted that the dimension of our physical spacetime is four. However, in some physical theories, other dimensions have been considered. Kaluza-Klein theories<sup>3)</sup> and superstring theories<sup>4)</sup> involve integer dimensions larger than four. In the dimensional regularization procedure<sup>5)</sup> and the  $\epsilon$  expansion about 4 dimensions,<sup>6)</sup> nonintegral dimensions are used as a useful mathematical device without physical significance.

We are interested in theories given in fractal spacetime dimensions<sup>7)</sup> or generalized Hausdorff dimensions,<sup>8)</sup> because quantum electrodynamics (QED) in a spacetime with fewer than four dimensions would be free of ultraviolet divergences.<sup>8)</sup>

We may consider our spacetime to be of a fractal nature,<sup>9)</sup> with a non-integer fractal dimension. The most basic property of a fractal is perhaps its fractal dimension  $n$ ,<sup>7)</sup> which is derived from  $n = \lim_{\delta \rightarrow 0} \ln(k(\delta)) / \ln(\delta^{-1})$ . The quantity  $k(\delta)$  here is the minimum number of elemental cubes of side  $\delta$  needed to cover the fractal. Let us consider the Cantor set as a concrete example of a fractal space. The two-dimensional three-Cantor space has a fractal dimension  $n = \ln(3^2 - 1) / \ln 3 = 1.89 \dots$ . This result can be generalized to a  $D$ -dimensional  $N$ -Cantor space,<sup>10)</sup> which has the fractal dimension  $n = \ln(N^D - 1) / \ln N \simeq D - 1 / (N^D \ln N)$ .

With the above example in mind, in this paper, we assume the dimension of spacetime to be slightly smaller than four.<sup>8)</sup> Within the framework of QED, this dimension of spacetime can be determined by calculating three Feynman diagrams, that of the electron self-energy, the photon self-energy, and the vertex graph. In addition, we investigate the validity of the perturbative expansion and the positivity of the fine-structure constant.

Let us start with the self-energy of an electron to second order,

$$\Sigma^{(2)}(p; \alpha_G) = ie_R^2 \int \frac{d^n k}{(2\pi)^n} \left[ \frac{\gamma^\mu (\not{p} - \not{k} + m_R) \gamma_\mu}{\{(p-k)^2 - m_R^2 + i\epsilon'\}(k^2 + i\epsilon')} \right. \\ \left. - (1 - \alpha_G) \frac{\gamma^\nu (\not{p} - \not{k} + m_R) \gamma^\mu k_\mu k_\nu}{\{(p-k)^2 - m_R^2 + i\epsilon'\}(k^2 + i\epsilon')^2} \right], \quad (1)$$

where  $\alpha_G$  is the gauge parameter,  $k$  is the momentum of the internal photon, and  $R$  indicates renormalized quantities. The bare mass of the electron  $m_B$  and a change of its mass  $\delta_m$  are given by

$$m_B = m_R - \delta_m, \tag{2}$$

$$\delta_m = -\Sigma_1^{(2)}(p^2 = m_R^2; \alpha_G)m_R, \tag{3}$$

where the self-energy diagram is written as

$$\Sigma^{(2)}(p; \alpha_G) = m_R \Sigma_1^{(2)}(p; \alpha_G) + (\not{p} - m_R) \Sigma_2^{(2)}(p; \alpha_G). \tag{4}$$

If we wish to ascribe most of the mass of the electron to its self-energy,<sup>11)</sup> then according to Eqs. (2) and (3) we have

$$\Sigma_1^{(2)}(p^2 = m_R^2; \alpha_G) = -1. \tag{5}$$

Applying the techniques of the dimensional regularization scheme<sup>5)</sup> to Eq. (5), we obtain in  $n = 4 - \epsilon$  dimensions,

$$\frac{3\alpha_R \nu^{-\epsilon}}{4\pi} \left( \frac{2}{\epsilon} + 2C - \frac{8}{3} + \ln \frac{\nu^2}{m_R^2} \right) = 1, \tag{6}$$

where  $2C \equiv \ln 4\pi - \gamma + 4$ ,  $\nu$  is a mass scale, and  $\alpha_R \equiv e_R^2/4\pi$  is the fine-structure constant, which has dimension  $\nu^\epsilon$ . Note that we do not take the limit  $\epsilon \rightarrow 0$  in Eq. (6).

The anomalous magnetic moment of the electron is defined as

$$a_e \equiv \frac{1}{2}(g_e - 2) \equiv F_2(0), \tag{7}$$

where  $F_2(0)$  is the Pauli form factor related to the electron-photon-electron vertex. Using the same methods as above,  $a_e$  is evaluated to lowest order as

$$a_e = \frac{\alpha_R \nu^{-\epsilon}}{2\pi} (1 + C\epsilon), \tag{8}$$

where we have neglected an infrared divergence. If  $\Delta a = a_{\text{th}} - a_{\text{exp}}$  represents the difference between the theoretical result and the experimental result for the anomalous magnetic moment of the electron, and if this difference is induced by a non-zero value  $\epsilon$ , we obtain

$$\Delta a = \frac{1}{2\pi} \{ \alpha_{R4} - \alpha_R \nu^{-\epsilon} (1 + C\epsilon) \}, \tag{9}$$

where  $\alpha_{R4}$  is the dimensionless fine-structure constant in four-dimensional spacetime.

In our method of calculation, we can calculate the finite renormalization constants  $Z_1$ ,  $Z_2$  and  $Z_3$  to prove  $Z_1 = Z_2$ . Using this equality in the relation  $e_R = Z_1^{-1} Z_2 Z_3^{1/2} e_B$ , we find that  $e_R = Z_3^{1/2} e_B$ ; i.e.,

$$e_R^2 = \left\{ 1 - \frac{\alpha_R \nu^{-\epsilon}}{3\pi} \left( \frac{2}{\epsilon} + 2C - 4 + \ln \frac{\nu^2}{m_R^2} \right) \right\} e_B^2. \tag{10}$$

From Eqs. (6), (9) and (10) we have

$$\epsilon = \frac{3\alpha_R^4}{2\pi} \left\{ 1 - \left( \frac{2\pi\Delta a}{\alpha_R} \right)^2 \right\}, \tag{11}$$

$$\alpha_B = \frac{9}{5}\alpha_R, \tag{12}$$

$$\nu = \exp \left( C + \frac{4\pi^2\Delta a}{3\alpha_R^2} \right), \tag{13}$$

where we regard  $\alpha_R$  and  $\nu$  as dimensionless quantities (measured in units of the electron mass).

We are now in a position to investigate the self-consistency of our model and determine the value of  $\epsilon$ . Using the experimental data,<sup>12)</sup> which give  $|\Delta a| < 1.9 \times 10^{-10}$ , we find the numerical values of  $\epsilon$  and  $\nu$ :  $\epsilon = 3.5 \times 10^{-3}$ ,  $\nu = m_R e^{3.0}$ . Using Eqs. (11) and (13), the second term on the right-hand side of Eq. (10) satisfies the inequality

$$\frac{\alpha_R \nu^{-\epsilon}}{3\pi} \left( \frac{2}{\epsilon} + 2C - 4 + \ln \frac{\nu^2}{m_R^2} \right) \simeq \frac{4}{9}(1 - C\epsilon) < 1.$$

This inequality ensures the validity of the perturbative expansion and that  $e_R^2 > 0$ . Equation (12) is consistent with the standard result<sup>13)</sup>  $\alpha_R < \alpha_B$ , based on the vacuum polarization.

The above value of  $\epsilon$  is significantly larger than those obtained in Refs. 8) and 14). This discrepancy is discussed below. The contribution of  $\Delta a$  is negligibly small in obtaining the value of  $\epsilon$ , which is mainly determined by the fine-structure constant  $\alpha_R$ , because  $\Delta a$  itself has a very small value. However, if a non-zero value of  $\Delta a$  is experimentally confirmed, it might be the case that this non-zero value is a consequence of the deviation of the dimension of spacetime from four, since a non-zero  $\Delta a$  leads to a nonvanishing  $\epsilon$ , as we found from Eq. (9).

Let us clarify the role of the mass scale  $\nu$ . If we set  $\nu = 1$  in Eq. (9), as assumed in Ref. 8), we have  $\Delta a = -\alpha_R C \epsilon / 2\pi$ . This implies that a value of  $1/|\epsilon|$  is too large, and the perturbation expansion is impossible. We can thus say that the value of  $\nu$  cannot be set equal to 1 in order for our model to be consistent.

In conclusion, we have found that the dimension of spacetime can be slightly smaller than four within the framework of QED; i.e.,  $n = 4 - \epsilon \simeq 4 - 3.5 \times 10^{-3}$ . We have also shown that our model is free of ultraviolet divergences and that the perturbation expansion is consistent.

Let us discuss what the value of  $\epsilon$  implies. Our value of  $\epsilon$ , representing the deviation of the dimension of spacetime from four, is significantly larger than that obtained in Ref. 8), and further is many orders of magnitude greater than the upper bound of conjectured values of  $\epsilon$  derived from other physical consideration.<sup>14)</sup> It is possible, however, to have a consistent model in which, the dimensionality of spacetime changes from  $n = 4$  at large distances to  $n = 4 - \epsilon$  at short distances. It is possible that the derived values of  $\epsilon$  in Ref. 14) are valid at larger distances, and our value of  $\epsilon$  is valid at very short distances.

The physical picture concerning the variation of dimensionality with distance may be imagined as follows. We have demonstrated that the dimension of a fractal spacetime could be slightly smaller than four. Holes in space are characteristic of Cantor spacetime,<sup>10)</sup> which is regarded as an example of a fractal spacetime. Mandelbrot<sup>7)</sup> introduced the term “tremas” for holes of this sort. These tremas may cause the dimension of spacetime to be slightly smaller than four at short distances. The effect of tremas on spacetime’s dimension would disappear on larger distance scales, and the dimension would approach  $n = 4$ .

It is interesting to inquire into the nature of tremas. Several possible mechanisms for tremas are now suggested. Though many aspects of tremas still remain rather speculative, string theory or spacetime quantization may provide mechanisms to explain how tremas could be formed. As another possibility, a bound on the measurability of lengths may be responsible for the emergence of tremas. We believe that spacetime has a minimum length, which would seem to represent the fundamental properties of nature. At scales smaller than this minimum length, short-distance structure could not be probed, which would imply a kind of holes in spacetime.

In the context of our model, let us discuss the minimum length. Note that the deviation  $\epsilon$  is related to the ultraviolet cutoff scale  $\Lambda$  by  $1/\epsilon \sim \ln(\Lambda/m_R)$ , which yields the maximum energy  $\Lambda \sim m_R e^{1/\epsilon} = m_R \exp(2\pi/3\alpha_{R4})$ . Then, using the quantum-mechanical uncertainty relations  $\Delta x \Delta p \gtrsim 1$ , we arrive at the relation  $\Delta x \gtrsim (1/m_R) \exp(-2\pi/3\alpha_{R4})$ , which implies the existence of a minimum length. This minimum length has also been obtained as a cutoff radius<sup>11)</sup> in the Dirac hole theory, where the self-energy of the electron has been calculated.

It would be very difficult to explain from first principles how tremas could be formed. This lies beyond the scope of the present article.

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