

Equivalence of the Weyl Vacuum and Normally Ordered Vacuum in the Moyal Quantization

Takao KOIKAWA^{*)}

*School of Social Information Studies, Otsuma Women's University,
Tama 206-0035, Japan*

(Received January 4, 2002)

We study features of the vacuum of the harmonic oscillator in the Moyal quantization. Two vacua are defined, one with the normal ordering and the other with Weyl ordering. Their equivalence up to an overall constant factor is shown by using a differential equation satisfied by the normally ordered vacuum.

There has been great interest in another method of quantization, the Moyal quantization,¹⁾ especially in its relation to superstring theory. In a previous paper,²⁾ we studied the vacuum of the harmonic oscillator in the Moyal quantization. We can introduce a pair of new variables a and a^\dagger instead of the phase variables q and p , which satisfy the Moyal bracket commutation relation isomorphic to the operator commutation relation (CR hereafter), between the creation and the annihilation operators. The Hamiltonian is expressed in terms of a and a^\dagger with the star product between them. Once the vacuum is obtained, the \star -gen function belonging to the \star -gen value $(n + \frac{1}{2})\hbar$ can be obtained by successively multiplying a^\dagger from the left and a from the right n times on the vacuum.^{3),4)} In the Moyal quantization, it is natural to call the lowest energy level solution, which can be expressed in terms of the Weyl products of a and a^\dagger , the vacuum because it vanishes when it is multiplied by a from the left or by a^\dagger from the right. We call it the Weyl vacuum. There is another way of defining the vacuum. By analogy to the operator formalism, we can introduce the normal ordering of the star products. Using this, we can also define the vacuum that we call the normally ordered vacuum. The apparent difference between these two vacua raises a question: Are they identical or not? In Ref. 2), we expanded the two vacua in terms of $1/\hbar$ and compared them. When we reorder the Weyl ordered vacuum as the normally ordered vacuum, there appear infinities at a lower order in $1/\hbar$. As long as we use the expansion in $1/\hbar$ for the comparison, what we get is at most the result order by order. With this, it is difficult to conclude anything definite.

In this paper, we study the same subject from a different point of view. Instead of rewriting the vacuum function in terms of the star products of a and a^\dagger , as in Ref. 2), we attempt to rewrite the normally ordered vacuum into a function explicitly and then compare two vacua. In order to obtain the functional form of the normally ordered vacuum, we derive a differential equation satisfied by a class of functions including the normally ordered vacuum, and then solve it. The solution includes constants of integration together with a parameter t , which is introduced to obtain

^{*)} E-mail: koikawa@otsuma.ac.jp

the differential equation. These integral constants are determined by the boundary condition at $t = 0$. The value of the parameter is set so that the function coincides with the normally ordered vacuum. We thus determine the functional form of the normally ordered vacuum.

First, we give the definitions and notation of the Moyal star product. The star product of two functions $f = f(q, p)$ and $g = g(q, p)$ is defined by

$$f \star g = \exp \left[i \frac{\hbar}{2} \left(\frac{\partial}{\partial q} \frac{\partial}{\partial \tilde{p}} - \frac{\partial}{\partial p} \frac{\partial}{\partial \tilde{q}} \right) \right] f(\mathbf{x}) g(\tilde{\mathbf{x}})|_{\mathbf{x}=\tilde{\mathbf{x}}}, \quad (1)$$

where $\mathbf{x} = (q, p)$ and $\tilde{\mathbf{x}} = (\tilde{q}, \tilde{p})$ and they are set equal after the derivatives are taken.

In the analysis of the harmonic oscillator, it is more convenient to use the complex variable z and its complex conjugate variable \bar{z} instead of the phase variables q and p . We define z as $z = q + ip$. The formulas for these variables are as follows:

$$z \star f(z, \bar{z}) = \left(z + \hbar \frac{\partial}{\partial \bar{z}} \right) f(z, \bar{z}), \quad (2)$$

$$\bar{z} \star f(z, \bar{z}) = \left(\bar{z} - \hbar \frac{\partial}{\partial z} \right) f(z, \bar{z}), \quad (3)$$

$$f(z, \bar{z}) \star z = \left(z - \hbar \frac{\partial}{\partial \bar{z}} \right) f(z, \bar{z}), \quad (4)$$

$$f(z, \bar{z}) \star \bar{z} = \left(\bar{z} + \hbar \frac{\partial}{\partial z} \right) f(z, \bar{z}). \quad (5)$$

We denote the star product CR of $f(z, \bar{z})$ and $g(z, \bar{z})$ as

$$[f(z, \bar{z}), g(z, \bar{z})] = f(z, \bar{z}) \star g(z, \bar{z}) - g(z, \bar{z}) \star f(z, \bar{z}). \quad (6)$$

Using the above formulas, we obtain the star product CR of z and \bar{z} :

$$[z, \bar{z}] = 2\hbar. \quad (7)$$

Introducing the new variables a and a^\dagger by $a = z/\sqrt{2}$ and $a^\dagger = \bar{z}/\sqrt{2}$, the star product CR reads

$$[a, a^\dagger] = \hbar, \quad (8)$$

which shows that the star product CR of a and a^\dagger is isomorphic to the operator CR of the creation and annihilation operators.

The Hamiltonian that we consider is that of the harmonic oscillator,

$$H = \frac{p^2}{2m} + k \frac{q^2}{2}, \quad (9)$$

where we assume $m = k = 1$. This can be rewritten using the variables z and \bar{z} or a and a^\dagger as

$$H = \frac{1}{2} \bar{z} z = \frac{1}{2} (\bar{z} \star z + \hbar) = a^\dagger \star a + \frac{\hbar}{2}. \quad (10)$$

We obtain the \star -gen function f_n belonging to the eigenvalue E_n by solving the \star -gen value equation:⁴⁾

$$H(z, \bar{z}) \star f_n(z, \bar{z}) = E_n f_n(z, \bar{z}). \tag{11}$$

The \star -gen function f_n belonging to the energy level $(n + \frac{1}{2})\hbar$ ($n = 0, 1, 2, \dots$) is expressed by the Laguerre polynomial $L_n(x)$ as

$$f_n(H) = e^{-\frac{2H}{\hbar}} L_n(2H), \quad (n = 0, 1, 2, \dots) \tag{12}$$

where $H = (q^2 + p^2)/2$. The 0-th \star -gen function f_0 belonging to the lowest level is

$$f_0 = e^{-\frac{2H}{\hbar}} = e^{-\frac{\bar{z}z}{\hbar}}, \tag{13}$$

which can be regarded as the vacuum. Other states are constructed by operating with \bar{z} from the left and z from the right repeatedly.

When (13) is expanded, each term $(\bar{z}z)^n$ is expressed by using the Weyl ordering of the star products. The Weyl ordering was first introduced as a map from classical functions to operator products in the quantization. Here we discuss the Weyl ordering of the star products within the classical framework. We denote the sum of the possible permutations of the star products of n variables \bar{z} and n variables z by $\text{Perm}(\bar{z}^n, z^n)$. Dividing this by the number of terms, we define the Weyl ordering of the n variables \bar{z} and n variables z by

$$(\bar{z}^n z^n)_W = \frac{(n!)^2}{(2n)!} \text{Perm}(\bar{z}^n, z^n). \tag{14}$$

One of the simplest examples is in the case $n = 1$:

$$(\bar{z}z)_W = \frac{1}{2}(\bar{z} \star z + z \star \bar{z}). \tag{15}$$

In general, we can express the power of $\bar{z}z$ in terms of the Weyl ordered star products as

$$(\bar{z}z)^n = (\bar{z}^n z^n)_W. \tag{16}$$

We are thus able to express the vacuum in terms of the Weyl ordered star products. We call this the Weyl vacuum.

We can also consider another type of vacuum by introducing the normal ordering of the star products. This is given by

$$: e^{-\frac{1}{2\hbar} \bar{z} \star z} := e^{-\frac{a^\dagger \star a}{\hbar}} : \sim: e^{-\frac{H}{\hbar}} :, \tag{17}$$

where the double dots denote the normal ordering of the star products, which is defined, in analogy to the operator case, by putting all a^\dagger to the left of all a . We call this the normally ordered vacuum. Explicitly they are written as

$$: e^{-\frac{a^\dagger \star a}{\hbar}} := 1 + \left(-\frac{1}{\hbar}\right) a^\dagger \star a + \frac{1}{2!} \left(-\frac{1}{\hbar}\right)^2 a^\dagger \star a^\dagger \star a \star a + \dots \tag{18}$$

We now have the Weyl vacuum and the normally ordered vacuum. These two vacua appear to be different, and this is what motivates the present and the previous study. The imaginary part of Eq. (11) requires that the \star -gen function should be a function of the variable H . The function $\phi(H)$ representing the vacuum in the Moyal quantization is characterized by the condition

$$a \star \phi(H) = \phi(H) \star a^\dagger = 0. \tag{19}$$

When the vacuum is obtained, the functions belonging to the higher levels are obtained algebraically by making use of the star product CR (8) and the expression of the Hamiltonian in terms of a and a^\dagger in (10). It is easy to show that both vacua satisfy this vacuum condition.

The Weyl vacuum has an explicit functional form, while the normally ordered vacuum has an operator-like expression. Then, there can be two methods in making the comparison of the two vacua. One is to rewrite the function into an operator-like form by introducing the Weyl ordering, which we did in our previous paper on this subject.²⁾ The other is to rewrite the operator-like expression into a function, which we do here. Introducing a parameter t , we define $f = f(t, H)$ by

$$f(t, H) =: e^{t\bar{z}\star z} ;, \tag{20}$$

which satisfies the condition

$$f(0, H) = 1. \tag{21}$$

We also note that, when the parameter t is set to $-(1/2\hbar)$, it becomes identical to the normally ordered vacuum:

$$f\left(-\frac{1}{2\hbar}, H\right) =: e^{-\frac{a^\dagger + a}{\hbar}} :. \tag{22}$$

Differentiating $f(t, H)$ with respect to the variable t , we obtain

$$\frac{\partial f(t, H)}{\partial t} = \bar{z} \star f(t, H) \star z. \tag{23}$$

Using the formulas (3) and (4), we rewrite the equation as

$$\begin{aligned} \frac{\partial f(t, H)}{\partial t} = 2 \left\{ H f(t, H) - \left(\frac{\hbar}{2}\right) (f(t, H) + 2H f'(t, H)) \right. \\ \left. + \left(\frac{\hbar}{2}\right)^2 (f'(t, H) + H f''(t, H)) \right\}, \end{aligned} \tag{24}$$

where $f'(t, H) = \frac{\partial f(t, H)}{\partial H}$ and $f''(t, H) = \frac{\partial^2 f(t, H)}{\partial H^2}$.

The solution of (24) is given by

$$f(t, H) = \frac{C_2}{C_1 - 2t} e^{\left(\frac{2}{\hbar}\right) \left(1 + \left(\frac{2}{\hbar}\right) \frac{1}{C_1 - 2t}\right) H}, \tag{25}$$

where C_1 and C_2 are constants of integration. These constants are determined by using the condition (21) as

$$1 + \left(\frac{2}{\hbar}\right) \frac{1}{C_1} = 0, \tag{26}$$

$$\frac{C_2}{C_1} = 1. \tag{27}$$

We thus obtain

$$C_1 = C_2 = -\frac{2}{\hbar}. \tag{28}$$

Substituting these results into (25), we obtain f as

$$f(t, H) = \frac{1}{1 + \hbar t} e^{\frac{2t}{1 + \hbar t} H}. \tag{29}$$

Setting $t = -(1/2\hbar)$, which reduces f to the normally ordered vacuum, as we mentioned above (22), we obtain

$$f\left(-\frac{1}{2\hbar}, H\right) = 2e^{-\frac{2H}{\hbar}}, \tag{30}$$

which is identical to the Weyl vacuum.

In order to show the uniqueness of the solution, we note that Eq. (24) is linear in $f(t, H)$, and so any linear combination of solutions is also a solution to the equation. Suppose that there is a solution $g(t, H)$, which is assumed to be different from the solution (30). We further require that it should satisfy the vacuum conditions (19). By using the formulas (2) and (5), they can be rewritten as

$$zg(t, H) + \hbar \frac{\partial g(t, H)}{\partial \bar{z}} = \bar{z}g(t, H) - \hbar \frac{\partial g(t, H)}{\partial z} = 0. \tag{31}$$

These lead to a differential equation for $g(t, H)$,

$$\frac{\hbar}{2} \frac{\partial g(t, H)}{\partial H} + g(t, H) = 0, \tag{32}$$

which is solved to yield

$$g(t, H) \sim e^{-\frac{2H}{\hbar}}. \tag{33}$$

The H dependence of this solution is the same as that of the solution (30). Therefore this contradicts the assumption that $g(t, H)$ is different from $f(t, H)$. We thus conclude that the solution (30) is a unique vacuum solution to Eq. (24), and so the normally ordered vacuum is equivalent to the Weyl ordered vacuum up to an overall constant factor.

When we approximate the Weyl ordered vacuum by expanding it in terms of $1/\hbar$ and reordering it into the normally ordered form, there appear infinities at a lower order in $1/\hbar$.²⁾ By taking the present result, the equivalence of the two vacua, into account, the appearance of these infinities suggests that they are factored out into

an overall constant when all of them are summed up. This elucidates the nature of the renormalization in the Moyal quantization.

One might think that the vacuum conditions (19) are sufficient to determine the functional form of the star product expressions. However, this is not completely true. It is true that they can determine the exponent of the exponential function when the star product expression is algebraically proven to be a function of H , but they cannot provide any information about the overall factor multiplied by the exponential function in which we are most interested. This should be contrasted with the differential equation (24), which determines both the exponent and the factor in front of the exponential function. In Ref. 2), we studied algebraically how infinities appear in reordering the Weyl ordered vacuum into a normally ordered form. In contrast with this, the solution (30) shows that the overall factor is finite for the normally ordered vacuum. This finiteness is important, because it can be a reference value for the renormalization of the Weyl ordered vacuum. If both had infinite factors, the idea of the renormalization would not make sense.

The method of obtaining the functional expression from the star product expression by using a differential equation is general and applicable to various expressions with star products. The straightforward extension of e^{tH} to a star product form is obtained by replacing H^n by the star products of n H . One might think that this may be another possibility for the vacuum. We examine this point by deriving the differential equation. Using the notation e_* , we define

$$e_*^{tH} = 1 + tH + \frac{1}{2!}t^2H \star H + \frac{1}{3!}t^3H \star H \star H + \dots, \tag{34}$$

where $H = z\bar{z}/2$. Since the star products of H are the function of the variable H , we denote e_*^{tH} as $F(t, H)$. In a manner similar to that used in the previous case, we can derive the differential equation satisfied by $F(t, H)$,

$$\frac{\partial F(t, H)}{\partial t} = H \star F(t, H) = HF(t, H) - \left(\frac{\hbar}{2}\right)^2 (F'(t, H) + HF''(t, H)), \tag{35}$$

where $F'_t(H) = \partial F(t, H)/\partial H$ and $F''_t(H) = \partial^2 F(t, H)/\partial H^2$. The solution is given by

$$F(t, H) = \operatorname{sech}\left(\frac{\hbar t}{2}\right) e^{\frac{2}{\hbar}(\tanh\frac{\hbar t}{2})H}. \tag{36}$$

Using $F(0, H) = 1$ and $\lim_{t \rightarrow \pm\infty} F(t, H) = 0$, we can show that

$$\begin{aligned} H \star \int_{-\infty}^0 F(t, H) dt &= \int_{-\infty}^0 \frac{\partial F(t, H)}{\partial t} dt = [F(t, H)]_{-\infty}^0 = 1, \\ H \star \left(-\int_0^{\infty} F(t, H) dt\right) &= -\int_0^{\infty} \frac{\partial F(t, H)}{\partial t} dt = -[F(t, H)]_0^{\infty} = 1. \end{aligned} \tag{37}$$

These results show that both $\int_{-\infty}^0 F(t, H) dt$ and $(-\int_0^{\infty} F(t, H) dt)$ are inverses of H with respect to the star product. We evaluate their difference to see whether they are identical. We have

$$\int_{-\infty}^0 F(t, H) dt - \left(-\int_0^{\infty} F(t, H) dt\right) = \int_{-\infty}^{\infty} F(t, H) dt = \frac{2\pi}{\hbar} J_0\left(-i\frac{2}{\hbar}\right), \tag{38}$$

where J_0 is the 0-th Bessel function of the first kind. This shows that the two inverses of H appearing in (37) are different, and thus the introduction of the finite integrals of e_\star^{tH} destroys the star product algebra. There is another pathological property of e_\star^{tH} . As in the previous case, one might think that we are able to show that (36) can be the vacuum by adjusting the value of t . However, this is not the case. Imposing the vacuum conditions (19), we can determine the value of t . Doing so, we obtain $\tanh(t\hbar/2) = -1$, with which $\operatorname{sech}(t\hbar/2)$ vanishes. Therefore, e_\star^{tH} cannot be the vacuum, even if we adjust the value of t .

The convergence of the star product expressions, in which an example similar to e_\star^{tH} can be found, is studied in Ref. 5). The convergence of an infinite series of star product expressions is a subtle problem. One typical example of such an infinite series is an exponential function with star products in the exponent. The infinities of the Weyl ordered vacuum and the present example of their reordering as the normally ordered expression might be related to this divergence problem of the infinite series. The normally ordered expression, $: e^{tH} :$, is free from pathological properties, unlike e_\star^{tH} . The normal ordering might represent a prescription that provides a tractable method to deal with the infinities of star product expressions related to the divergence problem appearing in infinite series.

The author would like to acknowledge the Doyou-kai members for useful discussions.

References

- 1) J. E. Moyal, Proc. Cambridge Phil. Soc. **45** (1949), 90.
- 2) T. Koikawa, Prog. Theor. Phys. **106** (2001), 1027.
- 3) T. Curtright, D. Fairlie and C. Zachos, Phys. Rev. D **58** (1998), 025002-1.
- 4) D. Fairlie, Proc. Cambridge Philos. Soc. **60** (1964), 581.
D. Fairlie and C. Mnogue, J. of Phys. A **24** (1991), 3807.
- 5) Y. Maeda, *Proceedings of Workshop on Noncommutative Geometry in String Theory and Field Theories*, ed. H. Ishikawa, M. Kato, T. Kawano, R. Sasaki, S. Terashima and S. Wata-mura (2001), p. 37.