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Dynamical Ordering of Periodic Accelerator Modes in the Standard Mapping

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In the standard mapping, period-q (≥ 1) accelerator modes appear through saddle-node bifurcations just after the destruction of the last KAM (Kolmogorov-Arnold-Moser) curve. Using the orbital symmetry resulting from reversibility, we derive the dynamical ordering for the orbits of symmetric accelerator modes.

§1. Introduction

In the standard mapping, the existence of a particular type of orbits referred to as accelerator modes (AM) plays an essential role in the anomalous transport in a stochastic layer.[1)−6)] The self-similar structure around the point of an AM and its effect on the diffusion process have also been investigated.[7)−9)] Accelerator modes appear through saddle-node bifurcations. In this work, we derive the appearance order, or so-called dynamical ordering, for AMs. The relation we obtain should be important for the study of diffusion in the standard mapping. We show that AMs appear as soon as the last KAM curve disintegrates.

The standard mapping T is defined in a cylinder ($y \in \boldsymbol{R}^1, x \in \boldsymbol{S}^1,\ \boldsymbol{S}^1 \times \boldsymbol{R}^1$) as

$$y_{n+1} = y_n + f(x_n), \tag{1}$$

$$x_{n+1} = x_n + y_{n+1} \ (\mathrm{Mod}\ 2\pi), \tag{2}$$

where $f(x) = a \sin x$, and a is a positive parameter. There are two fixed points, $P = (0, 0)$ and $Q = (\pi, 0)$, where P is a saddle, and Q is an elliptic point for $0 < a < 4$ and a saddle with reflection for $a > 4$.

Let $O(p_0) = \{p_0, p_1, \cdots\}$ be a positive orbit in (the universal cover of) the cylinder of a point p_0 with $p_i = T^i p_0$ and $p_i = (x_i, y_i)$. This positive orbit is constructed from the forward images of p_0. We say an orbit is a *period-q* (≥ 1) *accelerator mode* (q-AM) if this (positive) orbit satisfies the conditions $x_q - x_0 = 2\pi m_x$ and $y_q - y_0 = 2\pi m_y$ (where m_x and m_y are positive integers). Such an orbit jumps by $2\pi m_y$ in the y direction every q iterations, and the number of rotations around the cylinder increases due to this increment in the value of the y coordinate. Thus the accelerator mode is not a true periodic orbit in a cylinder. However after applying the operation $\mathrm{Mod}(y_i, 2\pi)$ to every point $p_i \in O(p_0)$ in the cylinder, we can regard it as a periodic orbit. Hereafter T is assumed to be defined in the torus ($x \in \boldsymbol{S}^1, y \in \boldsymbol{S}^1$).

In §2, the basic tools for our study are introduced. Theorems on the dynam-

ical ordering are proved and a relation regarding the appearance of AMs and the destruction of the last KAM curve is obtained in §3.

§2. Preliminaries

In a torus, two involutions of T $(= H \circ G)$ can be expressed by

$$H\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} -y \\ x-y \end{pmatrix} \quad \text{and} \quad G\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} -y-f(x) \\ x \end{pmatrix}. \tag{3}$$

The sets of fixed points of H and G are called the symmetry axes.[10),11)] In the following, we employ the universal cover $\boldsymbol{R}^1 \times \boldsymbol{R}^1$ of the torus and attach the symbol $\widehat{\ }$ to the mapping function T and other quantities.

In the universal cover, there are infinitely many symmetry axes, satisfying

$$\begin{array}{lll} \widehat{S}_1(k) : y = 2k\pi & \text{for} & \widehat{H}, \\ \widehat{S}_2(k) : y = k\pi - f(x)/2 & \text{for} & \widehat{G}, \end{array} \tag{4}$$

where k is an integer. We restrict our attention to q-AMs having two points on the symmetry axes, and call them q-SAMs. Let $\widehat{I} = \{(\widehat{x}, \widehat{y}) \in \widehat{S}_1(0) | 0 < \widehat{x} < \pi\}$ and $\widehat{J} = \{(\widehat{x}, \widehat{y}) \in \widehat{S}_2(0) | 0 < \widehat{x} < \pi\}$.

From this point, we consider q-SAMs starting in $\widehat{I}$ and those starting in $\widehat{J}$. We use the expression $q_{m_x}^{m_y}$ for a q-SAM starting in $\widehat{I}$ or $\widehat{J}$, where $2\pi m_x$ and $2\pi m_y$ are the distances the orbit traverses in the x- and y-directions during the first q iterations.

Let $W_u^1(\widehat{P}_{i,j})$ and $W_u^2(\widehat{P}_{i,j})$ be branches of the unstable manifold emanating from saddle $\widehat{P}_{i,j} = (2\pi i, 2\pi j)$ in the upper-right and lower-left directions. Let $W_s^1(\widehat{P}_{i,j})$ and $W_s^2(\widehat{P}_{i,j})$ be branches of the stable manifold approaching $\widehat{P}_{i,j}$ from the upper-left and lower-right directions. Let $\hat{u}$ be the first intersection point of $W_u^1(\widehat{P}_{0,0})$ with $x = \pi$ and $\hat{v}$ be the first intersection point of $W_s^1(\widehat{P}_{1,0})$ with $y = 2(x - \pi)$. $\widehat{u}$ and $\widehat{v}$ are primary homoclinic points. Let $\widehat{\gamma}_u$ be the arc of $W_u^1(\widehat{P}_{0,0})$ connecting $\widehat{T}^{-1}\hat{v}$ and $\hat{u}$, and let $\widehat{\gamma}_s$ be the arc of $W_s^1(\widehat{P}_{1,0})$ connecting $\widehat{T}^{-1}\widehat{v}$ and $\widehat{u}$. The homoclinic lobe $\widehat{U}$ is defined as the open area bounded by $\widehat{\gamma}_u$ and $\widehat{\gamma}_s$. We define the open intervals $\widehat{I}_i \in \widehat{I}$ and $\widehat{J}_i \in \widehat{J}$ for $i \geq 0$ as

$$\widehat{I}_i = \widehat{T}^{-i}\widehat{U} \cap \widehat{I}, \tag{5}$$

$$\widehat{J}_i = \widehat{T}^{-i}\widehat{U} \cap \widehat{J}. \tag{6}$$

In Fig. 1, the stable and unstable manifolds, the homoclinic lobe $\widehat{U}$, two symmetry axes $\widehat{S}_1(0)$ and $\widehat{S}_2(0)$, and several intervals are displayed.

We write $\widehat{I}_i \to \widehat{I}_j$ if the existence of interval $\widehat{I}_i$ implies the existence of $\widehat{I}_j$. According to the Lambda Lemma,[12)] we have the following order relations for intervals $\widehat{I}_i$ and $\widehat{J}_i$:

$$\widehat{I}_0 \to \widehat{I}_1 \to \widehat{I}_2 \to \cdots, \tag{7}$$

$$\widehat{J}_0 \to \widehat{J}_1 \to \widehat{J}_2 \to \cdots. \tag{8}$$

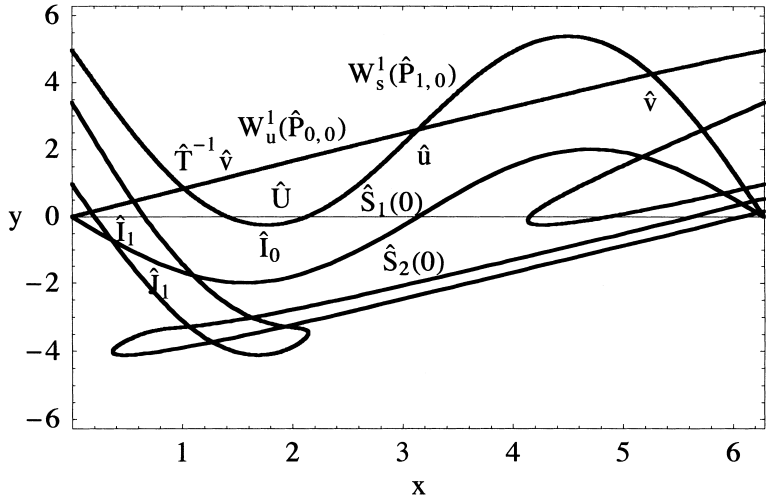

Fig. 1. The stable and unstable manifolds, the homoclinic lobe $\widehat{U}$, two symmetry axes $\widehat{S}_1(0)$ and $\widehat{S}_2(0)$, and the intervals $\widehat{I}_0$, $\widehat{I}_1$ and $\widehat{J}_1$ at $a = 4$.

Simple geometrical consideration leads us to the relation

$$\widehat{J}_i \to \widehat{I}_i. \tag{9}$$

Let $\widehat{A}_i$ and $\widehat{B}_i$ be two edge points of $\widehat{I}_i$. These are located in $W_s^1(\widehat{P}_{1,0})$. For $\widehat{T}^{i+k}\widehat{A}_i$ and $\widehat{T}^{i+k}\widehat{B}_i$ $(k \geq 1)$, the following holds.

Proposition 1. For $k \geq 1$, the two points $\widehat{T}^{i+k}\widehat{A}_i$ and $\widehat{T}^{i+k}\widehat{B}_i$ remain below $\widehat{S}_2(1)$.

Proof. The mapped points $\widehat{T}^{i+k}\widehat{A}_i$ and $\widehat{T}^{i+k}\widehat{B}_i$ are in $(\hat{v}, \widehat{P}_{1,0})_{W_s^1(\widehat{P}_{1,0})}$, the arc of $W_s^1(\widehat{P}_{1,0})$ connecting $\hat{v}$ and $\widehat{P}_{1,0}$. This arc, in turn, is in $\pi < \widehat{x} < 2\pi$ and $\widehat{y} > 0$. In the following, we prove that $(\hat{v}, \widehat{P}_{1,0})_{W_s^1(\widehat{P}_{1,0})} \cap \widehat{S}_2(1) = \emptyset$ holds.

If $\hat{v} \in \widehat{S}_2(1)$, then $W_u^2(\widehat{P}_{1,1})$ and $W_s^1(\widehat{P}_{1,0})$ intersect at $\hat{v}$. However, this is impossible, because $\hat{v}$ is the intersection of $W_u^1(\widehat{P}_{0,0})$ and $W_s^1(\widehat{P}_{1,0})$. Hence $\hat{v} \notin \widehat{S}_2(1)$. In the limit $a \to 0$, the $\widehat{y}$ coordinate of $\hat{v}$ is of order $\sqrt{a}$ $(\ll \pi)$, and thus $\hat{v}$ is below $\widehat{S}_2(1)$. The position of $\hat{v}$ changes continuously as a increases. Therefore $\hat{v}$ remains below $\widehat{S}_2(1)$ for any value of $a > 0$.

Next, we show that $(\hat{v}, \widehat{P}_{1,0})_{W_s^1(\widehat{P}_{1,0})}$ and $\widehat{S}_2(1)$ never intersect, and this completes the proof. Suppose the contrary, and let $\widehat{s}_1$ and $\widehat{s}_2$ be adjacent intersection points. These are located in the region satisfying $3\pi/2 < \widehat{x} < 2\pi, \widehat{y} > \pi$. The slope of $W_s^1(\widehat{P}_{1,0})$ is negative and finite.[13)] Due to reversibility, $W_s^1(\widehat{P}_{1,0})$ and $W_u^2(\widehat{P}_{1,1})$ intersect at $\widehat{s}_1$ and $\widehat{s}_2$. The slope of $(s_i, \widehat{P}_{1,1})_{W_u^2(\widehat{P}_{1,1})}$ is positive and finite.[13)] It is then easily confirmed that there is a point of negative or zero slope in $(\widehat{s}_1, \widehat{s}_2)_{W_u^2(\widehat{P}_{1,1})}$ or $(\widehat{s}_2, \widehat{s}_1)_{W_u^2(\widehat{P}_{1,1})}$. This is a contradiction. We can derive the same contradiction for the tangential situation with $s_1 = s_2$. (Q.E.D.)

§3. Dynamical ordering

3.1. *Several properties*

Let us summarize as the following properties that will later be used to prove our theorems.

Property 1. Let $O(\widehat{p}_0) = \{\widehat{p}_0, \widehat{p}_1, \cdots, \widehat{p}_{q-1}\}$ be the positive orbit of a q-SAM and assume that $\widehat{p}_0 = (\widehat{x}_0, \widehat{y}_0) \in \widehat{I}$ and $\hat{p}_q = (\widehat{x}_q, \widehat{y}_q) \in \widehat{S}_1(2m_y)$. The following relations hold:

[A1] $\widehat{y}_k = 2\pi m_y - \widehat{y}_{q-k}$. $(0 \le k \le q)$
[A2] $q = 2n + 1$. $(n \ge 0)$
[A3] $m_x = (n+1)m_y$.
[A4] $\hat{p}_n \in \widehat{S}_2(m_y)$.
[A5] $\hat{p}_{n-k} = \widehat{G}\hat{p}_{n+k}$. $(0 \le k \le n)$
[A6] $\hat{p}_k \in \widehat{U}$. $(0 \le k \le (q-1))$

Property 2. Let $O(\widehat{p}_0) = \{\widehat{p}_0, \widehat{p}_1, \cdots, \widehat{p}_{q-1}\}$ be the positive orbit of a q-SAM and assume that $\widehat{p}_0 = (\widehat{x}_0, \widehat{y}_0) \in \widehat{J}$ and $\hat{p}_q = (\widehat{x}_q, \widehat{y}_q) \in \widehat{S}_1(2m_y)$. The following relations hold:

[B1] $\widehat{y}_k = 2\pi m_y - \widehat{y}_{q-k+1}$. $(0 \le k \le q)$
[B2] $q = 2n$. $(n \ge 1)$
[B3] $m_x = nm_y$.
[B4] $\hat{p}_n \in \widehat{S}_2(m_y)$.
[B5] $\hat{p}_{n-k} = \widehat{G}\hat{p}_{n+k}$. $(0 \le k \le n)$
[B6] $\hat{p}_k \in \widehat{U}$. $(0 \le k \le (q-1))$

The above properties have been obtained by several authors.[1)–3)] Here we simply state the origin or meaning of [A1]–[A6]. [A1] is derived by simply applying the mapping. [A2] and [A3] are obtained from [A1]. [A2] means that one point of a SAM with odd period exists in $\widehat{I}_i$. Applying a theorem proven by de Vogelaere,[5)] [A4] and [A5] are derived. [A6] means that an orbit passes through $\widehat{U}$ to escape toward the region $y \to \infty$. Note $\widehat{U}$ is called a turnstile.[14)] We do not comment on [B1]–[B6].

[A4] and [A5] (resp., [B4] and [B5]) imply the following proposition.

Proposition 2. If $\hat{p}_n$ is located in $\widehat{S}_2(m_y)$ and $\hat{p}_{n-1}$ is located below $\widehat{S}_2(m_y)$, $\hat{p}_{n+1}$ is located above it.

3.2. *Theorems*

The following theorems determine the dynamical ordering of SAMs. In the following, the symbols $\downarrow$ and $\leftarrow$ have the same meaning as $\rightarrow$ in the expression of dynamical ordering.

Theorem 1. For SAMs starting in $\widehat{I}_i$ $(i \ge 0)$, the following dynamical ordering holds:

$$
\begin{array}{lccccccccc}
\widehat{I}_0: & 1^{m_y}_{m_y} & \to & 3^{m_y}_{2m_y} & \to & 5^{m_y}_{3m_y} & \to & 7^{m_y}_{4m_y} & \to \cdots \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\widehat{I}_1: & 3^{m_y}_{2m_y} & \to & 5^{m_y}_{3m_y} & \to & 7^{m_y}_{4m_y} & \to & 9^{m_y}_{5m_y} & \to \cdots \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\widehat{I}_2: & 5^{m_y}_{3m_y} & \to & 7^{m_y}_{4m_y} & \to & 9^{m_y}_{5m_y} & \to & 11^{m_y}_{6m_y} & \to \cdots \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots & \cdots & & \cdots & & \cdots & & \cdots &
\end{array}
$$

Here, $m_y \geq 1$.

Proof. According to [A2], SAMs with odd period appear in $\hat{I}_i$ $(i \geq 0)$. The relation between two suffices is determined by [A3]. Let us derive the relations for the first row with $m_y = 1$ and the first column with $m_y = 1$. Other cases can be treated similarly.

Let us consider the situation that 1^1_1 exists and $\widehat{T}\widehat{I}_0$ is tangent to or intersects $\widehat{S}_1(1)$. From this fact and Proposition 1, we have $\widehat{T}\widehat{I}_0 \cap \widehat{S}_2(1) \neq \emptyset$, because p_1 of 1^1_1 is located above $\widehat{S}_2(1)$ and the two ends of $\widehat{T}\widehat{I}_0$ are located below $\widehat{S}_2(1)$. The intersection points are the points $\hat{p}_1$ of 3^1_2, i.e. $1^1_1 \to 3^1_2$ (see Fig. 2). Next, let us consider the situation that 3^1_2 exists and $\widehat{T}^2 I_0$ is tangent to or intersects $\widehat{S}_2(1)$. $\hat{p}_1$ of 3^1_2 is located below $\widehat{S}_2(1)$, and $\hat{p}_2$ of 3^1_2 is located above $S_2(1)$ by Proposition 2. This implies that $\widehat{T}^2\widehat{I}_0 \cap \widehat{S}_2(1) \neq \emptyset$, i.e., $3^1_2 \to 5^1_3$ (see Fig. 3). Repeating this procedure, the order relations of the first row are obtained.

The arc $\widehat{I}_i$ and the stable manifold between A_i and B_i form a closed curve. Let us denote by $\widehat{D}_i$ the closed area bounded by these curves. Obviously, $\widehat{T}\widehat{D}_i \supset \widehat{D}_{i-1}$. Therefore, $\widehat{T}^{k+i}\widehat{D}_i \cap \widehat{S}_2(1) \neq \emptyset$ implies $\widehat{T}^{k+i+1}\widehat{D}_{i+1} \cap \widehat{S}_2(1) \neq \emptyset$. Thus, if $\widehat{T}^{i+1}\widehat{I}_i \cap \widehat{S}_2(1) \neq \emptyset$, which implies the existence of $(2i+1)^1_{i+1}$, then $\widehat{T}^{i+2}\widehat{I}_{i+1} \cap \widehat{S}_2(1) \neq \emptyset$ holds (see Fig. 4). The intersection is a point of $(2i+3)^1_{i+2}$. (Q.E.D.)

Theorem 2. For SAMs starting in $\widehat{J}_i$ $(i \geq 0)$, the following dynamical ordering holds:

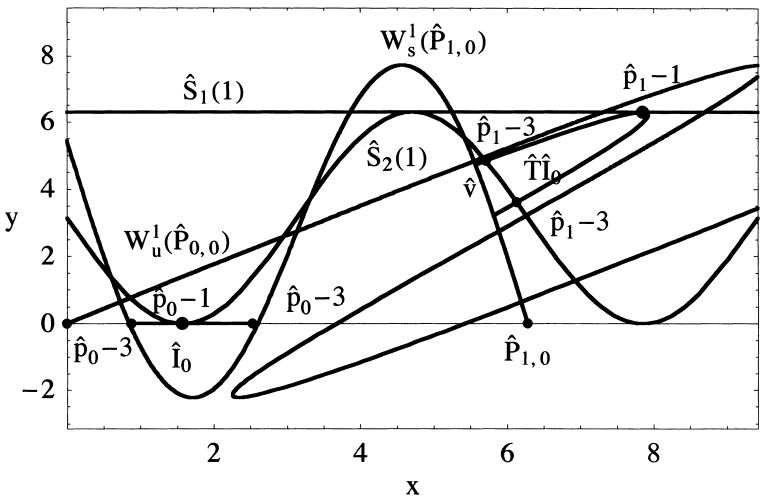

Fig. 2. The situation with $\widehat{T}\widehat{I}_0 \cap \widehat{S}_2(1) \neq \emptyset$ at $a = 2\pi$. The expression $\hat{p}_i - k$ indicates a point of $\hat{p}_i$ of the k-SAM.

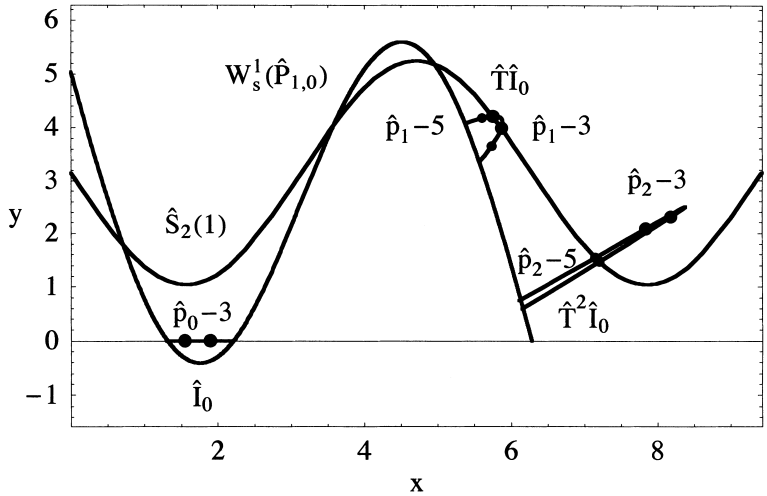

Fig. 3. $\widehat{T}\widehat{I}_0 \cap \widehat{S}_2(1) \neq \emptyset$ implies $\widehat{T}^2\widehat{I}_0 \cap \widehat{S}_2(1) \neq \emptyset$ $(a = 4.2)$.

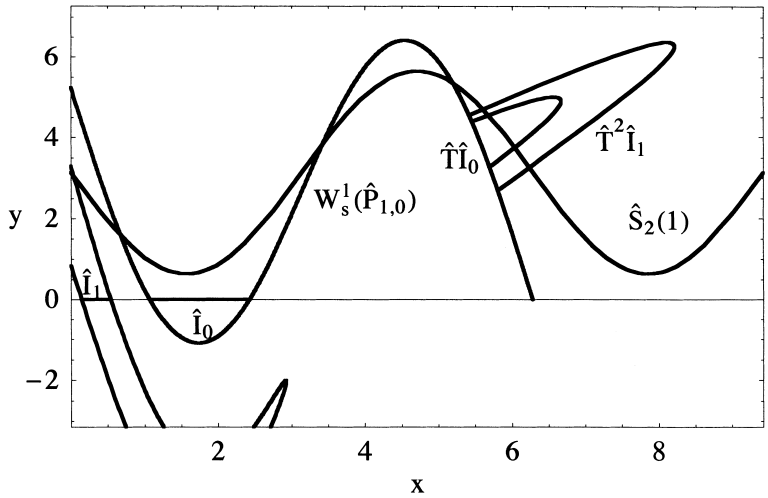

Fig. 4. $\widehat{T}\widehat{I}_0 \cap \widehat{S}_2(1) \neq \emptyset$ implies $\widehat{T}^2\widehat{I}_1 \cap \widehat{S}_2(1) \neq \emptyset$ $(a = 5)$.

$$
\begin{array}{lccccccccc}
\widehat{J}_0: & 1_{m_y}^{m_y} & \to & 2_{m_y}^{m_y} & \to & 4_{2m_y}^{m_y} & \to & 6_{3m_y}^{m_y} & \to \cdots \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\widehat{J}_1: & 2_{m_y}^{m_y} & \to & 4_{2m_y}^{m_y} & \to & 6_{3m_y}^{m_y} & \to & 8_{4m_y}^{m_y} & \to \cdots \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\widehat{J}_2: & 4_{2m_y}^{m_y} & \to & 6_{3m_y}^{m_y} & \to & 8_{4m_y}^{m_y} & \to & 10_{5m_y}^{m_y} & \to \cdots \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots & \cdots & & \cdots & & \cdots & & \cdots &
\end{array}
$$

Here, $m_y \geq 1$.

Proof. The proof of Theorem 2 is similar to that of Theorem 1 and therefore omitted. (Q.E.D.)

Adopting matrix notation, we write the (i, j) element of the dynamical ordering

in Theorem 1 as $DO_o^{m_y}(i,j)$ $(i \geq 0, j \geq 1)$. We also use $DO_e^{m_y}(i,j)$ for that in Theorem 2. For example, $DO_o^1(0,1)$ is the SAM 3_2^1 in Theorem 1. The suffices o and e stand for 'odd' and 'even'.

Theorem 3.
i) $DO_o^{m_y+1}(i,j) \to DO_o^{m_y}(i,j)$, and $DO_e^{m_y+1}(i,j) \to DO_e^{m_y}(i,j)$.
ii) $DO_e^{m_y}(i,j) \to DO_o^{m_y}(i,j)$.

Proof. As a typical example of $DO_o^{m_y+1}(i,j) \to DO_o^{m_y}(i,j)$, we prove the following ordering:

$$\widehat{I}_0 : \ 3_2^1 \leftarrow 3_4^2 \leftarrow 3_6^3 \leftarrow \cdots . \tag{10}$$

Figure 5 displays the situation in which 3_4^2 exists, where $\widehat{T}\widehat{I}_0 \cap \widehat{S}_2(2) \neq \emptyset$ holds. Because $\widehat{S}_2(1)$ remains below $\widehat{S}_2(2)$, $\widehat{T}\widehat{I}_0 \cap \widehat{S}_2(1) \neq \emptyset$ holds. This implies the relation $3_2^1 \leftarrow 3_4^2$. Repeating this procedure, Eq. (10) is proved. The proofs for the other cases are similar and therefore omitted.

Next, we prove the relation

$$2_1^1 \in \widehat{J}_0 \to 3_2^1 \in \widehat{I}_0, \tag{11}$$

as an example of relations ii). The proofs for the other cases are similar and therefore omitted. Figure 6 depicts the situation in which $2_1^1 \in \widehat{J}_0$ exists, where $\widehat{T}\widehat{J}_0 \cap \widehat{S}_2(1) \neq \emptyset$ holds. $\widehat{T}\widehat{I}_0$ and the arc of $W_s^1(\widehat{P}_{1,0})$ form an area that contains the area bounded by $\widehat{T}\widehat{J}_0$ and the arc of $W_s^1(\widehat{P}_{1,0})$. Thus $\widehat{T}\widehat{I}_0 \cap \widehat{S}_2(1) \neq \emptyset$ holds. This implies the existence of a point of 3_2^1 in $\widehat{I}_0$. (Q.E.D.)

Definition. A SAM is called a GSAM if it has either the following symmetry property (a) or (b).
(a) For odd period $q = 2n+1$, the points p_k $(0 \leq k \leq n-1)$ remain below $S_2(m_y)$ and p_k $(n+1 \leq k \leq 2n)$ remain above it.

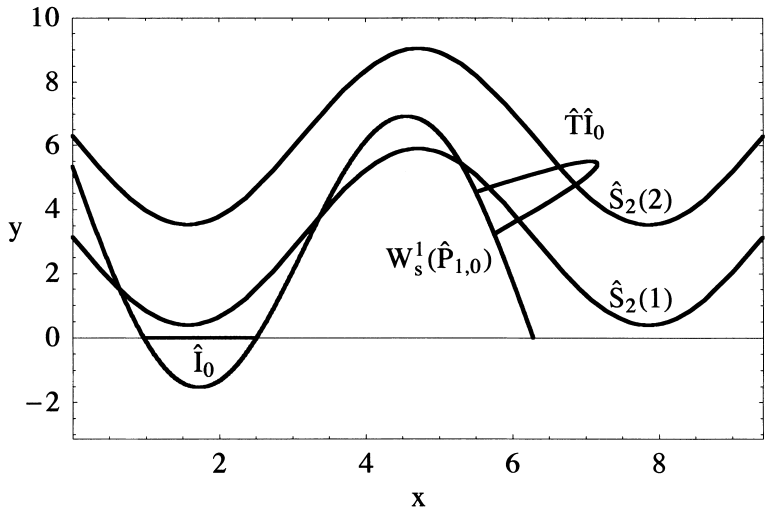

Fig. 5. If $\widehat{T}\widehat{I}_0 \cap \widehat{S}_2(2) \neq \emptyset$ holds, then $\widehat{T}\widehat{I}_0 \cap \widehat{S}_2(1) \neq \emptyset$ holds ($a = 5.5$).

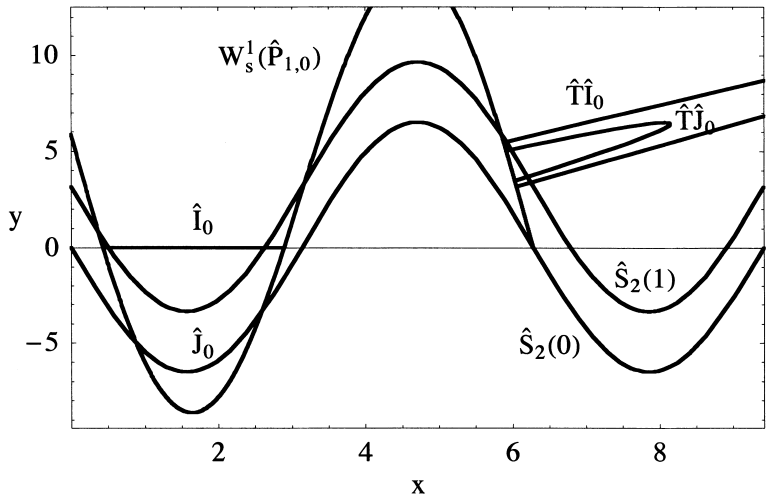

Fig. 6. If $\widehat{T}\widehat{J}_0 \cap \widehat{S}_2(1) \neq \emptyset$ holds, then $\widehat{T}\widehat{I}_0 \cap \widehat{S}_2(1) \neq \emptyset$ holds ($a = 13$).

(b) For even period $q = 2n$, the points p_k $(0 \leq k \leq n-1)$ remain below $S_2(m_y)$ and p_k $(n+1 \leq k \leq 2n-1)$ remain above it.

Corollary. Every SAM in Theorems 1 and 2 is a GSAM. In each interval I_i or J_i, a GSAM appears before other SAMs.

Proof. We prove the cases in Theorem 1. There is a parameter value for which $(2n+1)^{m_y}_{(n+1)m_y} \in \widehat{I}_0$ exists but SAMs with period less than $(2n+1)$ do not exist in $\widehat{I}_0$. For this parameter value, $\widehat{T}^k\widehat{I}_0 \cap \widehat{S}_2(m_y) = \emptyset$ $(0 \leq k \leq n-1)$; that is, all $\widehat{T}^k\widehat{I}_0$ $(0 \leq k \leq n-1)$ are located below $\widehat{S}_2(m_y)$, and thus all the points $\widehat{p}_0, \cdots, \widehat{p}_{n-1}$ are below it. The first statement of the theorem follows. This also means that the GSAM is the first SAM in each interval. The proofs for the other cases are similar and therefore omitted. (Q.E.D.)

According to the above Corollary, if we find a point of a SAM in some I_i or J_i, we can find a point of a GSAM with the same period in the same interval. This fact is used in the next subsection.

3.3. *Relation of the last KAM curve and SAMs*

We now introduce several critical values in order to study the relation between the destruction of the last KAM curve and the appearance of SAMs. Let $a^o(i,j)$ be the critical parameter value at which a GSAM of the (i,j)-element in Theorem 1 with $m_y = 1$ appears through a saddle-node bifurcation.

Let $a_c(j)$ $(j \geq 1)$ be the critical parameter value at which $\widehat{T}^{j-1}\gamma_u$ is tangent to $\widehat{S}_2(1)$. Let $a_c(\widehat{I}_i)$ be the critical parameter value at which the interval $\widehat{I}_i$ $(i \geq 0)$ appears. Let a_c be the critical value at which the last KAM curve is destroyed. The numerical value of a_c found by Greene[15)] is $0.9716\cdots$, though Greene's residue method has not been fully verified.

Theorem 4.

$$\lim_{i\to\infty} a^o(i,j) = a_c(j), \tag{12}$$

$$\lim_{j\to\infty} a_c(j) = a_c, \tag{13}$$

$$\lim_{j\to\infty} a^o(i,j) = \max(a_c(\widehat{I}_i), a_c). \tag{14}$$

Proof of Eq. (12). We show that for a given j and for any i, we have $a^o(i,j) > a_c(j)$. At $a = a_c(j)$, the closure of $\widehat{T}^{j-1}\widehat{U}$ is tangent to $\widehat{S}_2(1)$, and all $\widehat{T}^{k-1}\widehat{U}(1 \le k < j)$ are below $\widehat{S}_2(1)$. Therefore there is no GSAM of the (i,j)-element in Theorem 1 with $m_y = 1$. This implies $a^o(i,j) > a_c(j)$.

Next, let us consider a parameter $a = a_c(j)+\varepsilon$ for any $\varepsilon > 0$. For this parameter, $\widehat{T}^{j-1}\widehat{\gamma}_u \cap \widehat{S}_2(1) \neq \emptyset$. By the Lambda Lemma, there is a sequence of intervals $T^{i+j-1}\widehat{I}_i$ that accumulates at $\widehat{T}^{j-1}\widehat{\gamma}_u$ as $i \to \infty$. Hence, there exists an i such that $\widehat{T}^{i+j-1}\widehat{I}_i \cap \widehat{S}_2(1) \neq \emptyset$, which implies the existence of a GSAM of the (i,j)-element, that is, $a^o(i,j) \le a_c(j) + \varepsilon$. Since ε is arbitrary, we obtain Eq. (12). (Q.E.D.)

Lemma 1. There exists a GSAM starting in $S_1(0)$ for any $a > a_c$.
Proof. We first consider the mapping in a cylinder. Let $V = \{(x,y) : y \le 0\}$ and define the set D by

$$D = \bigcup_{k\ge 0}^{\infty} T^k V. \tag{15}$$

As shown by a theorem by Birkhoff,[16] D is not bounded in the y direction if there is no KAM curve encircling the cylinder. Thus there exists a positive integer n such that $T^n S_1(0) \cap S_2(1) \neq \emptyset$ holds for $a > a_c$. Now we return to the torus. The intersection is a point of a SAM with $q = 2n+1$. If this is a GSAM, the proof is complete. If it is not a GSAM, there exists a GSAM, as follows from the Corollary. (Q.E.D.)

Lemma 2. In a cylinder, $W_u^1(P)$ intersects $S_2(1)$ for any $a > a_c$.
Proof. For convenience, we denote a saddle at $(2\pi, 0)$ as P'. Let Γ be $[P,v]_{W_u^1(P)} \cup [v,P']_{W_s^1(P')}$, where v is defined in §2. Γ is a Jordan curve dividing the cylinder into two regions. Using the same argument as in the proof of Lemma 1 and the fact that $T^n[v,P']_{W_s^1(P')}$ $(n \to \infty)$ shrinks toward P', the lemma is proved.[17] (Q.E.D.)

Proof of Eq. (13). Trivially, $a_c(j) > a_c$ for any j. By Lemma 1, we obtain Eq. (13). (Q.E.D.)

Proof of Eq. (14). From Theorem 1, we see that $a^o(i,j)$ is a strictly decreasing function of j. There are two cases: $a_c(\widehat{I}_i) \ge a_c$ and $a_c(\widehat{I}_i) < a_c$. In the first case, $a^o(i,j)$ accumulates at $a_c(\widehat{I}_i)$ in the limit $j \to \infty$, as follows from Lemma 1. In the second case, a branch of $\widehat{T}^m\widehat{I}_i$ $(i < m \to \infty)$ for any i accumulates at $W_u^1(\widehat{P})$, as follows from the Lambda Lemma. From this fact and Lemma 2, there exists a positive integer n $(\gg i)$ such that $\widehat{T}^n\widehat{I}_i \cap \widehat{S}_2(1) \neq \emptyset$. From this fact and the

Corollary, there exists a point of a GSAM with odd period $(2n+1)$ in $\widehat{I}_i$, and $a^o(i,j)$ accumulates at a_c in the limit $j \to \infty$. (Q.E.D.)

The same relations hold for GSAMs of Theorem 2. We plot the stable and unstable manifolds numerically (see Fig. 1) and observe the tangency situation between $W_s(\widehat{P}_{1,0})$ and the symmetry axis. These numerical calculations show that $a_c(\widehat{I}_0) \simeq 3.68 > a_c(\widehat{I}_1) \simeq 1.67 > a_c(\widehat{I}_2) \simeq 1.09 > a_c > a_c(\widehat{I}_3) \simeq 0.83$, and $a_c(\widehat{J}_0) \simeq 8.27 > a_c(\widehat{J}_1) \simeq 2.30 > a_c(\widehat{J}_2) \simeq 1.32 > a_c > a_c(\widehat{J}_3) \simeq 0.94$.

Finally we comment on the effect of GSAMs on anomalous diffusion. The big islands of a short periodic GSAM (for example, a period-3 GSAM) cause the anomalous diffusion. The islands of long periodic GSAMs are extremely small. Just after the destruction of the last KAM curve, infinitely many GSAMs with long periods appear. The contribution from such GSAMs to the anomalous diffusion may be negligibly small. Therefore the observation in Ref. 4) that the diffusion around a_c is normal may be due to a lack of accuracy. Carrying out a detailed numerical estimation of the anomalous diffusion is a task that remains.

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