

## Non-Symmetric Periodic Points in Reversible Maps

— Examples from the Standard Map —

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The existence of non-symmetric non-Birkhoff periodic points and non-symmetric periodic points of the accelerator mode in standard-like maps is proved. Positions of these points are approximately determined for large parameter values. The number of such points is shown to diverge as the parameter value goes to infinity. We have found two routes for the appearance of non-symmetric periodic points. One is the equi-period bifurcation of a symmetric periodic point and the other is simultaneous saddle-node bifurcations. These two bifurcations seem to disprove the necessity of ‘hidden symmetry’ introduced by Murakami et al.(2001). We do not know whether or not these are the only routes for the appearance of these points. Numerical examples are considered for the standard map.

### §1. Introduction

Let us consider an area-preserving reversible map<sup>1)-3)</sup>  $T$  of an orientable surface such as  $\mathbf{R}^2$ ,  $\mathbf{S}^1 \times \mathbf{R}$ , or  $\mathbf{S}^2$ . The smoothness of the map is of  $C^1$  class. By definition,  $T$  is expressed in the form

$$T = H \circ G, \quad (1)$$

with involutions  $H$  and  $G$ , i.e.,  $H \circ H = G \circ G = \text{Id}$  and  $\det \nabla H = \det \nabla G = -1$ .

Following DeVogelaere,<sup>1)</sup> we introduce the operators

$$M_n = T^n G \quad \text{for } n \in \mathbf{Z} \quad (2)$$

and invariant sets

$$\mathcal{M}_n = \{r | r = M_n r\} \quad \text{for } n \in \mathbf{Z}. \quad (3)$$

$\mathcal{M}_0$  is the set of points invariant under  $G$  and  $\mathcal{M}_1$  is the set of points invariant under  $H$ . Both  $\mathcal{M}_0$  and  $\mathcal{M}_1$  are called *symmetry axes*. We have  $\mathcal{M}_{2n} = T^n \mathcal{M}_0$  and  $\mathcal{M}_{2n+1} = T^n \mathcal{M}_1$ ; that is, we obtain  $\mathcal{M}_n$  as iterates of  $\mathcal{M}_0$  or  $\mathcal{M}_1$ . We also have  $\mathcal{M}_{-n} = G \mathcal{M}_n$ . For  $n > m$ , define

$$\mathcal{M}_{n,m} = \mathcal{M}_n \cap \mathcal{M}_m. \quad (4)$$

$\mathcal{M}_n$  is called a set of symmetric points and  $\mathcal{M}_{n,m}$  is called a set of *doubly symmetric points*. A point is doubly symmetric if two points of its orbit are in the set of doubly symmetric points. For more details, see DeVogelaere.<sup>1)</sup>

**Theorem 1** (Ref. 1)). A point is symmetric periodic if and only if it is doubly symmetric.

This theorem works well when we wish to find symmetric periodic points. In that case, we iterate a symmetry axis and look for an intersection with another symmetry axis. The intersection points are symmetric periodic points. The orbit of a symmetric periodic point will be called a symmetric periodic orbit.

**Theorem 2** (Ref. 1)). There are three kinds of symmetric periodic points.

- i)  $r$  is  $2k$ -periodic,  $r \in \mathcal{M}_0$  and  $T^k r \in \mathcal{M}_0$ .
- ii)  $r$  is  $2k$ -periodic,  $r \in \mathcal{M}_1$  and  $T^k r \in \mathcal{M}_1$ .
- iii)  $r$  is  $(2k - 1)$ -periodic,  $r \in \mathcal{M}_0$  and  $T^{k-1} r \in \mathcal{M}_{-1}$ .

The purpose of the present paper is to prove the existence of non-symmetric periodic points in standard-like maps (see §2) and to give examples from the standard map. Theorem 1 in its present form does not provide us useful information when we look for non-symmetric periodic points, because it does not characterize non-symmetric points. For this reason, we seek to express Theorem 1 in a different form.

Let  $r_0$  be a periodic point of period  $q$  and  $A = \{r_0, r_1, \dots, r_{q-1}\}$  be its orbit such that  $r_i = Tr_{i-1}$  for  $1 \leq i \leq q - 1$  and  $r_0 = Tr_{q-1}$ . We define a set  $GA$  by  $GA = \{s_0, s_1, \dots, s_{q-1}\}$  with  $s_i = Gr_i$ .

**Property 1.** If  $A$  is a periodic orbit of period  $q$ , then  $GA$  is also a periodic orbit of the same period.

*Proof.* First, we have  $s_0 = Gr_0 = GT^q r_0 = T^{-q} Gr_0 = T^{-q} s_0$ ; i.e.,  $s_0$  is a periodic orbit of period  $q$  or its divisor. Then,  $s_i = Gr_i = GT^i r_0 = T^{-i} Gr_0 = T^{-i} s_0$  for  $1 \leq i < q$  which implies that  $s_i, i = 1, \dots, q - 1$  are all different as long as  $r_i, i = 1, \dots, q - 1$  are all distinct. Q.E.D.

**Corollary to Theorem 1.** The periodic orbit  $A$  is symmetric if and only if  $A = GA$ .

*Proof.* (Sufficiency) We know from Property 1 that  $GA$  is a periodic orbit. Hence, we need only show that some  $s_i$  coincides with some  $r_j$ . Let us consider case i) of Theorem 2. In this case,  $s_0 = Gr_0 = r_0$ . In case ii) of Theorem 2,  $s_0 = Gr_0 = GTr_0 = T^{-1} r_0 = r_{2k-1}$ . In case iii) of Theorem 2,  $s_0 = Gr_0 = r_0$ . Thus in any case,  $A = GA$ .

(Necessity) Because  $A = GA$ , we at least have  $s_i = r_0$  for some  $0 \leq i \leq q - 1$ . Then  $r_0 = s_i = Gr_i = GT^i r_0 = T^{-i} Gr_0 \in \mathcal{M}_{-i}$ ; i.e.,  $r_0$  is symmetric. Therefore, by Theorem 1,  $A$  is symmetric. (Q.E.D.)

**Remarks.** i) We considered  $A$  and  $GA$  above. Here let us consider  $HA = \{Hr_0, Hr_1, \dots, Hr_{q-1}\}$ . We have  $Hr_i = HT^i r_0 = HTT^{i-1} r_0 = Gr_{i-1} = s_{i-1}$ , and hence we have  $GA = HA$ .

- ii) If a periodic orbit satisfies  $A \neq GA$ , then  $A$  and  $GA$  are non-symmetric orbits.

§2. Non-symmetric non-Birkhoff periodic points for the standard map

Let us consider a reversible monotone twist map<sup>2)</sup>  $T$  on the cylinder  $S^1 \times R$

$$T : \begin{cases} y_{n+1} = y_n + f(x_n), \\ x_{n+1} = x_n + y_{n+1} \pmod{2\pi}, \end{cases} \tag{5}$$

where  $f(x)$  is an odd function of period  $2\pi$ . The function  $f(x)$  has at least one zero in  $(0, 2\pi)$ . We assume that  $f(x)$  has only one zero at  $x = \pi$  in this interval. In such a situation, the map (5) is sometimes referred to as a “standard-like map” or an “extended standard map”. It is to be noted that this map is doubly reversible;<sup>4)</sup> that is, the map has two forms of expression (1),

$$H \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} y \\ y - x \end{pmatrix} \quad \text{and} \quad G \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} y + f(x) \\ -x \end{pmatrix}, \tag{6}$$

and

$$\tilde{H} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} -y \\ x - y \end{pmatrix} \quad \text{and} \quad \tilde{G} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} -y - f(x) \\ x \end{pmatrix}, \tag{7}$$

where the  $x$ -coordinate is taken modulo  $2\pi$ .

**Property 2.** The stabilities of the periodic orbits  $A$  and  $GA$  are identical.

*Proof.* Let  $J_i$  be the linear map from  $r_i = (y_i, x_i)$  to  $r_{i+1} = (y_{i+1}, x_{i+1})$ .  $J_i$  has the form

$$J_i = \begin{pmatrix} 1 & a_i \\ 1 & 1 + a_i \end{pmatrix}, \tag{8}$$

where we write  $a_i = f'(x_i)$ . The linear map for one period of  $A$  is written  $J_q J_{q-1} \cdots J_1$ . Next, let  $\tilde{J}_i$  be the linear map from  $s_i$  to  $s_{i-1}$ . The linear map for one period of  $GA$  is written  $\tilde{J}_1 \cdots \tilde{J}_{q-1} \tilde{J}_q$ . Because of the fact that  $s_i = Gr_i = (y_i + f(x_i), -x_i)$ , the linear map depends only on the  $x$ -coordinates of  $s_i$ , and  $f'(x)$  is an even function, we have  $\tilde{J}_i = J_i$ . Therefore, the linear map for one period of  $GA$  is  $J_1 J_2 \cdots J_q$ . Proposition 1 below ensures that  $\text{trace}(J_q J_{q-1} \cdots J_1) = \text{trace}(J_1 \cdots J_{q-1} J_q)$ . Q.E.D.

Let us introduce two matrices  $M^{(n)}$  and  $N^{(n)}$  for  $n \geq 1$  by

$$\begin{aligned} M^{(n)} &= J_n J_{n-1} \cdots J_1 \equiv \begin{pmatrix} A_1^{(n)} & A_2^{(n)} \\ A_3^{(n)} & A_4^{(n)} \end{pmatrix}, \\ N^{(n)} &= J_1 J_2 \cdots J_n \equiv \begin{pmatrix} B_1^{(n)} & B_2^{(n)} \\ B_3^{(n)} & B_4^{(n)} \end{pmatrix}, \end{aligned} \tag{9}$$

where  $J_i$  is given by Eq. (8).

**Proposition 1.** We have the following equalities for the linear maps (9) of the periodic orbit for the standard-like map (5):

$$\text{i) } A_3^{(n)} + A_4^{(n)} = B_3^{(n)} + B_4^{(n)},$$

- ii)  $A_2^{(n)} = B_2^{(n)}$ ,
- iii)  $A_1^{(n)} + A_2^{(n)} = B_4^{(n)}$ , and
- iv)  $B_1^{(n)} + B_2^{(n)} = A_4^{(n)}$ .

*Proof.* We use induction. For  $k = 1$  and  $k = 2$ , we easily confirm the above equalities. Then, assuming the above four equalities hold for  $k = n$ , by direct computation we find that they also hold for  $k = n + 1$ . Q.E.D.

**Remark.** In which class of reversible maps does property 2 hold? This may be an interesting question.

2.1. *Period-3*

In this and the following sections, we set  $f(x) = a \sin x$ ; i.e., we consider the so-called standard map.

We look for non-symmetric non-monotone<sup>2),3)</sup> (or non-Birkhoff) periodic orbits with rotation number  $1/3$  for the reversibility expressed in Eq. (6). We consider an orbit starting at  $p_0(x_0, y_0)$ . The general conditions that  $p_0$  be a periodic point of period 3 with rotation number  $1/3$  are

$$\sin x_0 - \sin(x_0 - y_0) = \frac{2\pi - 3y_0}{a} \tag{10}$$

and

$$\sin x_0 + \sin(x_0 - y_0) + \sin(x_0 + y_0 + a \sin x_0) = 0. \tag{11}$$

Postponing the determination of whether the point is symmetric and whether the orbit of the point is monotone, we first look for the position of  $p_0$  and estimate the number of such periodic points for sufficiently large  $a > 0$ . Let us fix  $a$  to a sufficiently large value  $a \gg 1$ . In what follows, we write  $\varepsilon = 1/a \ll 1$  and for simplicity assume  $|y| \ll a$ . We use  $c_i, i = 1, 2, \dots$  to express quantities of order unity. From Eq. (10) omitting the suffix ‘0’, we obtain

$$\sin x - \sin(x - y) = c_1\varepsilon, \tag{12}$$

where  $c_1 < 0$  for  $3y > 2\pi$ , and  $c_1 \geq 0$  otherwise. The solution to this equation is

$$y = 2n\pi - c_2\varepsilon, \tag{13}$$

or

$$y = 2x + (2n - 1)\pi + c_2\varepsilon. \tag{14}$$

In the former case,  $|n| \ll a/6\pi$  should be satisfied, whereas in the latter case,  $|x + n\pi| \ll a/2$  should be satisfied.

Let us first consider the case (13). Substituting Eq. (13) into Eq. (11), we obtain for each  $n$

$$\sin(x + a \sin x) = -2 \sin x + c_3\varepsilon. \tag{15}$$

The absolute value of the right-hand side should not be greater than 1. Therefore,

$$-\pi/6 + 2m\pi + c_4\varepsilon < x < \pi/6 + 2m\pi - c_4\varepsilon, \tag{16}$$

or

$$5\pi/6 + 2m\pi + c_4\varepsilon < x < 7\pi/6 + 2m\pi - c_4\varepsilon. \tag{17}$$

Let us consider the first of the above intervals of  $x$  for  $m = 0$ . In this interval, the value of  $2 \sin x$  increases monotonically from  $-1$  to  $+1$ . Correspondingly the argument  $x + a \sin x$  of the function on the left-hand side increases from  $-a/2 - \pi/6$  to  $a/2 + \pi/6$ . Then the function  $\sin(x + a \sin x)$  oscillates approximately  $a/2\pi$  times between  $-1$  and  $+1$ . Equation (15) has at least two solutions in each oscillation. This implies that Eq. (15) has at least  $a/\pi$  solutions for  $-\pi/6 < x < \pi/6$ . The argument is the same for the range of  $x$  in Eq. (17).

Let us summarize the case of Eq. (13). For  $a \gg 1$ , for each  $n$  such that  $|n| \ll a/6\pi$ , there are approximately  $a/2\pi$  period-3 points of rotation number  $1/3$  in the region satisfying Eqs. (13) and (16) or Eqs. (13) and (17). In particular, the number of points diverges as  $a \rightarrow \infty$ .

Next, let us consider the case (14). Substituting Eq. (14) into (11) and omitting the suffix '0', we obtain for each  $n$

$$\sin(3x + a \sin x) = 2 \sin x + c_5\varepsilon. \tag{18}$$

Again,  $x$  should be contained in one of the intervals (16) or (17). The succeeding argument is similar to the above. For  $n = 0$ , the solutions are close to the arc

$$y = 2x + \pi, \quad -\pi/6 < x < \pi/6. \tag{19}$$

The number of solutions of (18) diverges as  $a \rightarrow \infty$ .

Before proceeding further, let us check the sign of  $c_2$  in Eqs. (13) and (14). For  $n \geq 1$ , substitution of (13) into (12) yields

$$\sin x - \sin(x + c_2\varepsilon) < 0. \tag{20}$$

$c_2$  is positive if  $x$  is in the range (16), whereas  $c_2$  is negative if  $x$  is in the range (17). For  $n = 0$ , we have  $\sin x - \sin(x + c_2\varepsilon) > 0$ . Therefore,  $c_2$  is negative if  $x$  is in the range (16), whereas  $c_2$  is positive if  $x$  is in the range (17). Let us next substitute (14) into (12) for  $n \geq -1$  and consider as representative the range (16) with  $m = 0, 2$  and the range (17) with  $m = 0$ . In these cases, we obtain  $\sin x - \sin(x + c_2\varepsilon) > 0$ .  $c_2$  is positive if  $x$  is in the range (16), whereas  $c_2$  is negative if  $x$  is in the range (17). For other values of  $n$  and  $m$ , we easily obtain the sign of  $c_2$ . The approximate positions of  $p_0(x_0, y_0)$  in a small neighborhood of the fundamental domain ( $0 \leq x < 2\pi$ ,  $0 \leq y \leq 2\pi$ ) are depicted in Fig. 1. From the figure, we see that these points are non-symmetric, except those on the line  $x = n\pi$  or on  $y = 2(x - n\pi)$ .

Now, let us determine whether or not orbits of these points are monotone (i.e., Birkhoff). We take a period-3 point that satisfies  $y_0 = 2\pi + c\varepsilon$  ( $c > 0$ ) with  $5\pi/6 < x_0 < 7\pi/6$ . We have

$$\begin{aligned} y_1 &= y_0 + a \sin x_0, \\ y_2 &= y_0 - a \sin(x_0 - c\varepsilon). \end{aligned} \tag{21}$$

The condition  $y_1 < 0$  is satisfied if  $\pi + 2\pi/a < x_0 < 7\pi/6$ , while the condition  $y_2 < 0$  is satisfied if  $5\pi/6 < x_0 < \pi - 2\pi/a$ . In other words, period-3 points are all

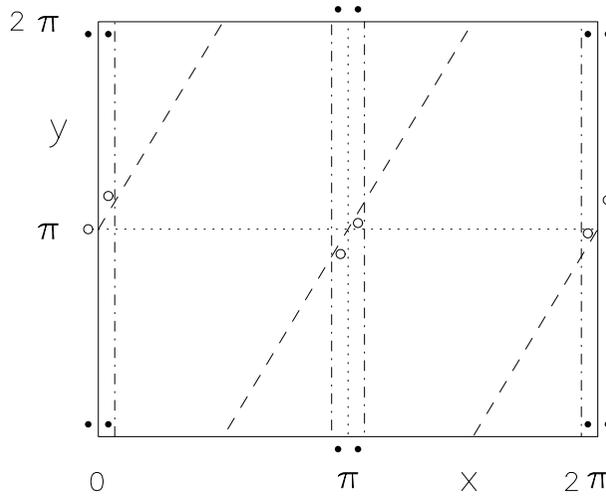


Fig. 1. Approximate positions of period-3 points for the standard map when the parameter is large enough. Periodic points satisfying Eqs. (13) and (11) are denoted by  $\bullet$ , whereas those satisfying Eqs. (14) and (11) are denoted by  $\circ$ .

non-Birkhoff, except the points in  $\pi - 2\pi/a < x < \pi + 2\pi/a$ . Next, consider a point in  $y_0 = 2\pi - c\varepsilon (c > 0)$ ,  $-\pi/6 < x_0 < \pi/6$ , while the condition  $y_1 < 0$  is satisfied if  $-\pi/6 < x_0 \leq -2\pi/a$ . The condition  $y_2 < 0$  is satisfied if  $2\pi/a \leq x_0 < \pi/6$ . Thus period-3 points in  $y_0 = 2\pi - c\varepsilon (c > 0)$  with  $-\pi/6 < x_0 < \pi/6$ , are non-Birkhoff except in the small region  $y_0 = 2\pi - c\varepsilon (c > 0)$  with  $-2\pi/a \leq x_0 \leq 2\pi/a$ . For other period-3 points in Fig. 1, the argument is similar. Our conclusion is that most period-3 points are non-Birkhoff. Birkhoff points remain extremely close to  $y = n\pi$ .

In the above, we restricted ourselves to  $|y| \ll a$ . But this restriction is not essential. For  $y \ll a$ , we can obtain a well-defined relation between  $y_0$  and  $x_0$  from (10), and substituting this relation into (11), we reach a conclusion similar to that.

Finally, let us briefly consider the case of small  $a > 0$ . In order for Eq. (10) to have a solution for  $a \rightarrow 0$ , we need to have  $3y_0 \rightarrow 2\pi$ . We easily confirm that such solutions tend to unique symmetric solutions that satisfy  $\sin x_0 = 0$ ,  $y_0 = (2/3)\pi$ ; that is, there are only symmetric periodic points for sufficiently small  $a > 0$ . To determine how non-symmetric periodic points bifurcate is a rather difficult problem, and therefore we have it as the subject of another paper.

### 2.2. Period-4 and higher periods

We look for non-symmetric non-Birkhoff periodic orbits with rotation number  $1/4$  with respect to the reversibility expressed in Eq. (6). The general conditions that  $p_0$  be a periodic point of period 4 with rotation number  $1/4$  are

$$2 \sin x_0 - \sin(x_0 - y_0) + \sin(x_0 + y_0 + a \sin x_0) = \frac{2\pi - 4y_0}{a}, \tag{22}$$

and

$$-\sin x_0 + 2 \sin(x_0 - y_0) + \sin(x_0 - 2y_0 + a \sin(x_0 - y_0)) = \frac{4y_0 - 2\pi}{a}. \tag{23}$$

In contrast to the case of period-3 points (see Eq. (10)), there are many relations between  $x_0$  and  $y_0$  that satisfy (22), and each of these together with (23), gives the same number of periodic points as the combination of (10) and (11). Therefore the number of period-4 points is substantially larger than that of period-3 points.

Now, let us consider the case  $2k\pi/a \leq \sin x_0 < 2(k+1)\pi/a \ll 1$ ,  $|y_0| \ll a$ , with  $a \gg 1$ . As before, we set  $\varepsilon = 1/a$  and use  $c_i, i = 1, 2, \dots$  as constants of order unity. For the value of  $x_0$  in this interval, Eq. (22) becomes

$$\sin(y_0 + a \sin x_0) = -\sin y_0 - c_1\varepsilon. \tag{24}$$

The solution to (24) is

$$\sin x = (2k + 1)\pi/a + c_3\varepsilon, \tag{25}$$

or

$$y = -(a/2) \sin x + n\pi + c_2\varepsilon, \quad 2k\pi/a \leq \sin x < 2(k + 1)\pi/a. \tag{26}$$

On substituting (25), Eq. (23) becomes

$$\sin(-2y_0 + a \sin(x_0 - y_0)) = 2 \sin y_0 + c_5\varepsilon. \tag{27}$$

In order for Eq. (27) to have a solution, either  $-\pi/6 + 2m\pi < y_0 < \pi/6 + 2m\pi$  or  $-5\pi/6 + 2m\pi < y_0 < 7\pi/6 + 2m\pi$  must hold. For this range of  $y_0$ ,  $a \sin(x_0 - y_0)$  moves from  $-a/2$  to  $+a/2$ , the left-hand side of (27) oscillates  $a/2\pi$  times, and the number of solutions to (27) is  $a/\pi$ . For each  $k$  it diverges as  $a \rightarrow \infty$ . Next, on substituting Eq. (26) into (23), we also obtain (27), with  $x_0$  and  $y_0$  satisfying (26).  $y_0$  must satisfy the same inequalities as in the above case. Again, the left-hand side of Eq. (27) oscillates  $a/2\pi$  times, hence the number of solutions is  $a/\pi$ .

Equations (25) and (26) indicate that most of the periodic points are not on the symmetry axes; i.e., they are non-symmetric periodic. Let us confirm the non-monotonicity of the corresponding periodic orbits. For simplicity, we only consider periodic orbits whose initial point  $p_0 = (x_0, y_0)$  is in  $2k\pi/a \leq \sin x_0 < 2(k+1)\pi/a$ ,  $k \geq 0$  and  $0 \leq y_0 < 2\pi$ .

In order to confirm the non-monotonicity of the corresponding periodic orbits, as before we estimate the sign of  $y_1, y_2, y_3 = y_{-1}$ . We have

$$\begin{aligned} y_1 &= y_0 + a \sin x_0, \\ y_2 &= y_0 + a \sin x_0 + a \sin(x_0 + y_0 + a \sin x_0), \\ y_{-1} &= y_0 - a \sin(x_0 - y_0). \end{aligned} \tag{28}$$

We wish to show either  $y_2 < 0$  or  $y_{-1} < 0$  for  $y_0 > 0$ . Using (22), we have  $y_2 = a \sin(x_0 - y_0) - a \sin x_0 + 2\pi - 3y_0$ . Condition  $y_2 \geq 0$  implies  $y_{-1} \leq 2\pi - a \sin x_0 - 2y_0 < 0$  for  $k \geq 1$ . For  $k = 0$ ,  $y_{-1}$  is negative if  $y_0 \geq \pi$ . We need to consider separately the cases in which  $0 < a \sin x_0 < 2\pi, 0 < y_0 < \pi$ . However, it seems that we cannot prove that periodic points in this case are non-monotone. In conclusion, possibly except the case  $k = 0$  for which  $0 < a \sin x_0 < 2\pi, 0 < y_0 < \pi$ , most period-4 orbits are non-symmetric and non-monotone.

For small  $a > 0$ , the comments of the last paragraph of §2.2 again apply although the positions of the periodic points are different.

Periodic points of rotation number  $1/q$ ,  $q > 4$ , can be obtained similarly. It is expected that the number of these points increases with their periods. Basically, these points are obtained geometrically as the crosspoints of a function with a horizontal line. The number of oscillations of the function increases as the period of points under consideration increases. The function is a sinusoidal function superposed with a sinusoidal function of frequency higher by a factor  $1/a$ . As the period increases, a sinusoidal function of frequency higher by a factor  $1/a^2$  is superposed, and so on. We met a similar situation in Ref. 9) (see Fig. 3 there) in a slightly different context.

2.3. *Periodic points of the accelerator mode*

In this subsection, the standard map is considered as defined on the torus. Periodic points of the accelerator mode (AM) are periodic on the torus, whereas they

are not periodic on the universal cover.<sup>5)–8)</sup> Using Expression (7), we look for period-3 points of accelerator modes.<sup>5)–7)</sup> We work in the universal cover  $\mathbf{R}^2$  of the torus, and we consider an AM with period-3 (hereafter referred to as a 3-AM) satisfying the conditions

$$\begin{cases} y_3 = y_0 + 2\pi, \\ x_3 = x_0 + 4\pi, \end{cases} \quad (29)$$

where  $p_i = (x_i, y_i) = T^i p_0$  and  $p_0$  is the initial point. The orbit starting at  $r_0 = H p_0$  is also a 3-AM.

The determining equations for the initial position of a period-3 orbit of accelerator mode are

$$3y_0 + a \sin x_0 - a \sin(x_0 - y_0) - 2\pi = 0, \quad (30)$$

and

$$\begin{aligned} & \sin x_0 + \sin(x_0 - y_0) \\ & + \sin(x_0 + y_0 + a \sin x_0) - 2\pi/a = 0. \end{aligned} \quad (31)$$

Conditions (30) and (31) are the same as conditions (10) and (11), except that a small term  $-2\pi/a$  is added to the second equation. Thus the solutions to (30) and (31) are different from those to (10) and (11), although their number and positions are almost the same. Most of them are not on the symmetry axes of involutions (7).

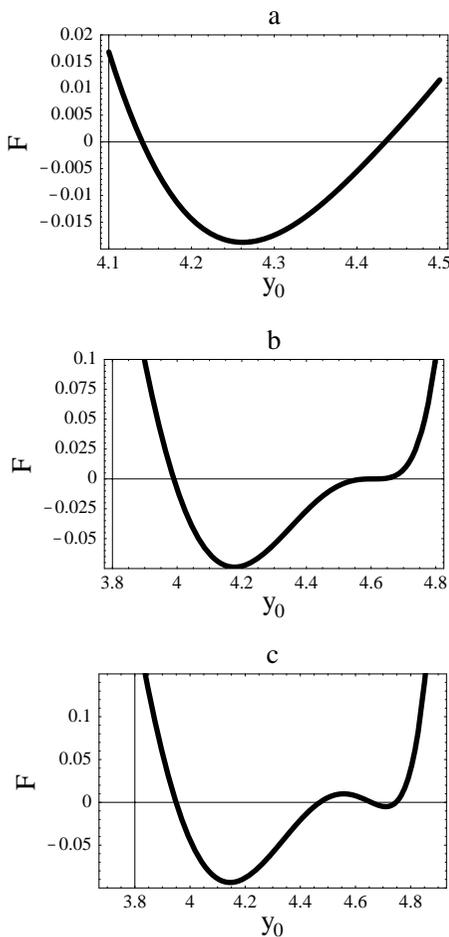


Fig. 2. The equi-period bifurcation of elliptic points. The ordinate is the determining function  $F$  for the position of  $y_0$ . Two elliptic points are already born in panel (c). (a)  $a = 7.3$ , (b)  $a = 7.581$  and (c)  $a = 7.7$ .

§3. Numerical examples

Our analysis in the preceding section has shown that for twist maps of the form (5) non-symmetric orbits always form a pair of the same stability; that is, if there is a non-symmetric orbit, then there necessarily exists a corresponding non-symmetric orbit of the same stability at a symmetric position with respect to involution  $G$  or  $H$ . Most of these orbits turn out to be non-Birkhoff. They appear through some kind of bifurcation, because non-Birkhoff orbits do not exist in an integrable system. They must appear in pairs.

Suppose that there are no non-symmetric periodic points. In the case of bifurcation with a mother periodic point, the only possibility is an *equi-period bifurcation* of symmetric periodic points. In the case of bifurcation without a mother periodic point, *simultaneous multiple tangent (saddle-node) bifurcations* are possible. Simultaneous saddle-node bifurcations at which  $4m$  ( $m \geq 1$ ) points appear are possible. In this case,  $m$  pairs of saddles and  $m$  pairs of elliptic points appear. If there already are non-symmetric points, the equi-period bifurcation of non-symmetric periodic points is added to the list of bifurcations. We have not yet detected such a bifurcation. We do not know if there is an additional kind of bifurcation.

Murakami et al.<sup>(6)</sup> observed in the standard map that a period-doubling bifurcation of a symmetric periodic orbit gives rise to a symmetric periodic orbit, whereas an equi-period bifurcation of a symmetric periodic orbit gives rise to non-symmetric periodic orbits, and in addition, these non-symmetric periodic orbits seemed to have no corresponding orbits obtained by reflection via involutions. They conjectured that this new ‘phenomenon’ is due to a ‘hidden symmetry’ of the standard map. Our analysis in §1, however, has shown that non-symmetric periodic orbits appear in pairs, although these pair orbits are sometimes far apart and difficult to be identified numerically. Therefore, it is not necessary to introduce any notion such as a ‘hidden symmetry’.

We first give numerical evidence of *equi-period bifurcation* of symmetric non-Birkhoff type orbits. Two points appear on the  $y$  axis at  $a = 7.221$ . Figure 2(a)

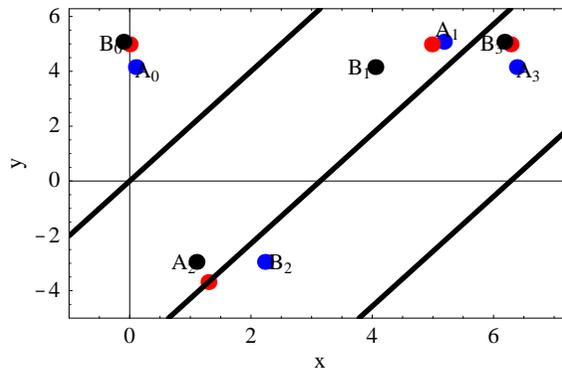


Fig. 3. Two NBOs  $\{A_0, A_1, A_2, A_3\}$  and  $\{B_0, B_1, B_2, B_3\}$  are bifurcated through the equi-period bifurcation ( $a = 9$ ).

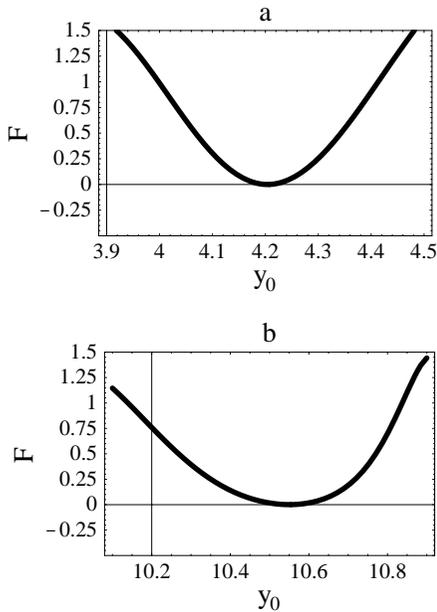


Fig. 4. The double saddle-node bifurcation at  $a = 19.13$ . The determining function is tangent to the horizontal axis at two different points.

shows that  $F(y_0) = 0$  has two solutions at  $a = 7.3$ . The left one is a saddle and the right one is an elliptic point. The elliptic point experiences an equi-period bifurcation at  $a = 7.581$  in Fig. 2(b), where it changes into a saddle and gives birth to two elliptic points, as in Fig. 2(c). The sum of Poincaré indices is conserved. Figure 3 shows the positions of a mother orbit and the bifurcated orbits.

Next, we give numerical evidence of a *double saddle-node bifurcation*. This occurs at  $a = 19.13$ . The determining function obtained from Eqs. (22) and (23) is tangent to the  $y_0$ -axis at  $y_0 = 4.18$  and  $10.6$  (see Fig. 4). The four bifurcated orbits are displayed in Fig. 5, where  $GA_0 = D_0$ ,  $GB_0 = C_0$ ,  $D_2 = HA_2$  and  $C_2 = HB_2$ . Further increasing the value of  $a$ , bifurcations of the same nature occur repeatedly. Thus, the number of non-symmetric and non-Birkhoff type periodic orbits is much

larger than the number of the symmetric non-Birkhoff ones in the limit  $a \rightarrow \infty$ .

Next, we show that a point of the 3-AM in the interval  $(0, \pi)$  of the  $x$ -axis appears via an equi-period bifurcation and two points  $A_0$  and  $B_0$  are bifurcated at  $a = 4.8219$ . Their orbits are displayed in Fig. 6 at  $a = 4.9$ , where  $B_0 = HA_0$  and  $B_1 = GA_1$  hold. The orbit starting from  $A_0$  or  $B_0$  does not pass through the symmetry axes.

Four 3-AMs appear through the double saddle-node bifurcations at  $a = 11.85$ .

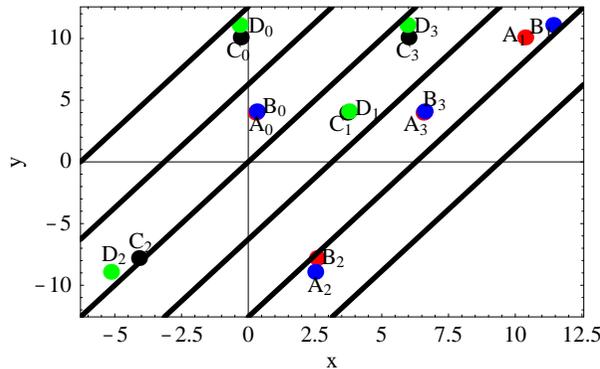


Fig. 5. Four NBOs bifurcated via the double saddle-node bifurcations ( $a = 22$ ).

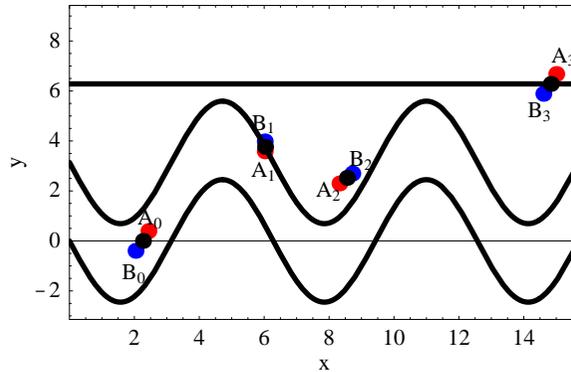


Fig. 6. Two 3-AMs bifurcated via the equi-period bifurcation of the symmetric one at  $a = 4.9$ .

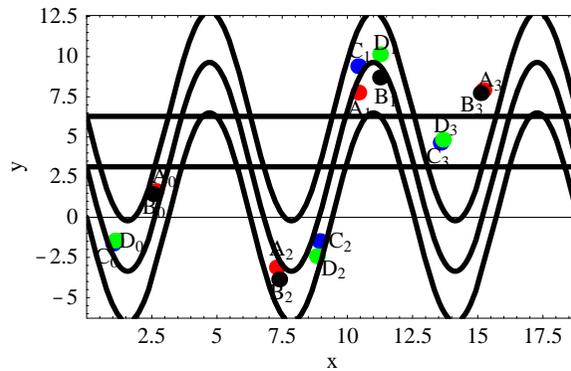


Fig. 7. Four 3-AMs bifurcated via the double saddle-node bifurcations at  $a = 13$ .

These orbits are displayed in Fig. 7 at  $a = 13$ , where  $D_0 = HA_0$ ,  $C_0 = HB_0$ ,  $D_1 = GA_1$  and  $C_1 = GB_1$  hold. These orbits do not pass through the symmetry axes.

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