

Multiplicity of Limit Cycle Attractors in Coupled Heteroclinic Cycles

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A square lattice distribution of coupled oscillators that have heteroclinic cycle attractors is studied. In this system, we find a novel type of patterns that is spatially disordered and periodic in time. These patterns are limit cycle attractors in the ambient phase space (i.e. not chaotic), and there exist many limit cycles, dividing the phase space into their basins. The patterns emerge from a local law concerning the difference between the phases of oscillators. The number of patterns grows exponentially as a function of the number of oscillators.

In recent decades, coupled oscillators have attracted much attention. They have been adopted as models of rhythmic and chaotic behavior of biological and other complex systems.^{1)–3)} Moreover, coupled oscillators themselves are thought to be important classes to investigate in the study of high-dimensional dynamical systems.^{2), 4), 5)} For analytical simplicity, limit cycles of a normal form type and phase oscillators have been frequently studied instead of detailed dynamics. While such studies have led to great development in the understanding of high-dimensional dynamical systems, there remain rich phenomena that cannot be described with such typical oscillators.

In this paper, we present a new class of coupled oscillators, in which each oscillator has a heteroclinic cycle attractor,⁶⁾ instead of a limit cycle or a phase oscillator. In contrast to the typical oscillators mentioned above, an oscillator with a heteroclinic cycle attractor has no characteristic time scale for the following reason. A heteroclinic cycle is composed of some saddle fixed points and heteroclinic orbits that connect the fixed points cyclically. When an orbit approaches a heteroclinic cycle attractor, it remains for an extended time in the neighborhood of fixed points and moves along with heteroclinic orbits quickly. The length of this staying time grows exponentially for each oscillation, while the speed with which it moves between fixed points changes little. Then, the period of oscillation grows exponentially, and hence the system has no characteristic time scale. This property of heteroclinic cycles can result in quite complex structures, such as the coexistence of infinitely many attractors.⁷⁾ Compared with the typical oscillators, this class of systems should provide another plentiful phenomena and allow our understandings of high-dimensional dynamical systems to become deeper.

In general dynamical systems, heteroclinic cycles are always structurally unstable.⁶⁾ However, if a system has certain constraints or symmetries, they can cause the appearance of invariant sets in the phase space, and a robust heteroclinic cycle can

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exist in the invariant sets. As a system with such constraints, we adopt a replicator system^{7)–11)} for each oscillator, given by

$$\frac{dx_i}{dt} = x_i \left(\sum_{j=1}^4 a_{ij} x_j - \sum_{j,k=1}^4 a_{jk} x_j x_k \right), \quad (i = 1, \dots, 4) \quad (1)$$

$$\sum_{i=1}^4 x_i = 1, \quad 0 \leq x_i \leq 1 \quad (2)$$

with the parameter matrix

$$(a_{ij}) = \begin{pmatrix} 0 & -2 & -1 & 1 \\ 1 & 0 & -2 & -1 \\ -1 & 1 & 0 & -2 \\ -2 & -1 & 1 & 0 \end{pmatrix}. \quad (3)$$

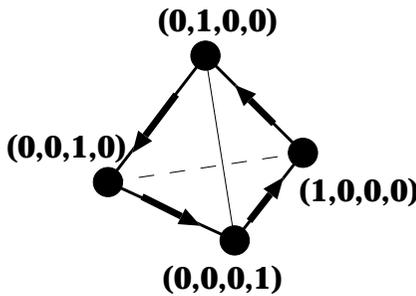


Fig. 1. A phase space of the replicator system with four components and a heteroclinic cycle. The filled circles represent saddle fixed points, and the thick solid lines with arrows represent heteroclinic orbits.

With the constraints (2), the phase space of the replicator system with four components is restricted to the region internal to a tetrahedron with vertices $\mathbf{x} = (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$ (Fig. 1). The vertices, edges ($\mathbf{x} | x_i + x_j = 1, x_k = x_l = 0$) and surfaces ($\mathbf{x} | x_i + x_j + x_k = 1, x_l = 0$) of the tetrahedron become invariant sets. In particular, vertices are always fixed points for any parameter matrix. Using the parameter matrix (3), an attracting heteroclinic cycle is constructed with the four vertices as saddle fixed points and the cyclically connecting edges as heteroclinic orbits (Fig. 1).

In our model, the replicator systems are distributed on a square lattice and nearest neighbors are coupled diffusively. The equations are given by

$$\begin{aligned} \frac{dx_i^{(u,v)}}{dt} &= x_i^{(u,v)} \left(\sum_{j=1}^4 a_{ij} x_j^{(u,v)} - \sum_{j,k=1}^4 a_{jk} x_j^{(u,v)} x_k^{(u,v)} \right) \\ &+ D(x_i^{(u-1,v)} + x_i^{(u,v-1)} + x_i^{(u+1,v)} + x_i^{(u,v+1)} - 4x_i^{(u,v)}), \end{aligned} \quad (4)$$

$(i = 1, \dots, 4)$

where (u, v) is a site index that represents the location of an oscillator, and D is the diffusion constant between adjoining sites. Free boundary conditions are employed. Because there is no non-diagonal diffusion term in the coupling method and the diffusion constant of each component has the same positive value, spatially uniform oscillation never becomes unstable.

A replicator system can be used as a model of an ecological system or a chemical reaction network of self-catalyzing molecules. Therefore, the situation described by

our system can occur in an ecological system on dotting sites (e.g. a system on trees in an orchard) or a population dynamical system consisting of self-catalyzing proteins in cells.

Before we state our main results for a system defined on a 2-dimensional array, we mention briefly our preliminary results for systems defined on 1-dimensional lattices for comparison. If we choose free boundary conditions for a 1-dimensional system (with ≥ 2 oscillators) all replicators are synchronized, and the only attractor corresponds to spatially uniform oscillation. Because no replicators are chaotic and we choose simple diffusive coupling as the interactions between oscillators, as mentioned above, this result is easily understood. If a system has periodic boundaries (with ≥ 3 oscillators) with a certain small diffusion constant, attractors corresponding to traveling waves and spatially uniform oscillation exist.

In this paper, we report the discovery of a novel class of patterns that are spatially disordered but periodic in time (Figs. 2 and 3). From different initial conditions, a large variety of patterns are observed (Fig. 2). Figure 3 shows that

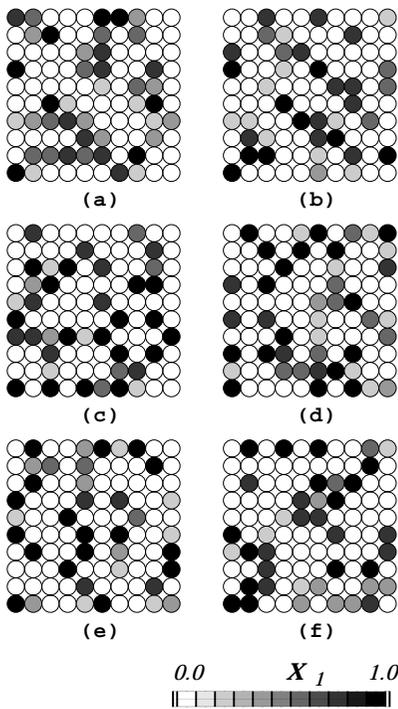


Fig. 2. 6 snapshots of distributions of $x_1^{(u,v)}$ on a gray scale with 10×10 sites and $D = 10^{-4}$. (a)–(e) were generated from different initial conditions, and they represent the configurations at the times that the (1,1)-oscillator (the bottom left site) realized to a given phase.

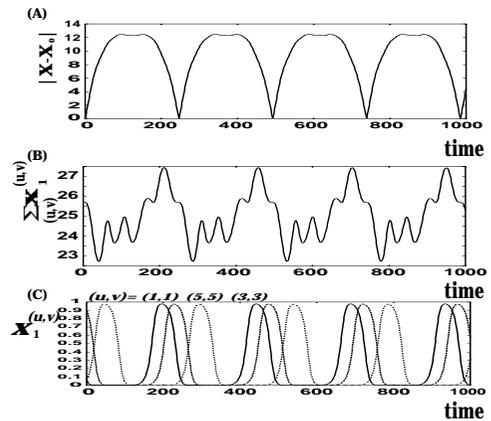


Fig. 3. Time series data corresponding to Fig. 2(e). (A): The distance between X_0 and X in the ambient phase space. Here, $X_0 = (x_1^{(1,1)}(t_0), x_2^{(1,1)}(t_0), \dots, x_3^{(10,10)}(t_0), x_4^{(10,10)}(t_0))$ is a point in a trajectory at a time t_0 (after approaching sufficiently close to an attractor), and X is the trajectory after this time. (B): The mean field oscillation of x_1 . (C): Oscillations of x_1 at the sites (1,1), (3,3) and (5,5). The exact recurrence in (A) indicates that the disordered pattern forms a limit cycle in the ambient phase space. Comparing (A) and (C), it is seen that the oscillators synchronize with the limit cycle.

these disordered patterns are stable limit cycle attractors in the ambient phase space. Therefore, these patterns are exactly recurrent and not chaotic. The different patterns correspond to different limit cycles, and they divide the phase space into their basins. After a system approaches sufficiently close to an attractor, the frequency of each oscillator synchronizes with the limit cycle (Fig. 3). This means that the frequencies of all oscillators become entrained, while the phases of the oscillations remain different. Disordered patterns are observed in the weak coupling range ($D \leq 0.01$), while rotating spiral patterns of a well-known type are seen with a certain large diffusion constant ($D \simeq 0.1$).

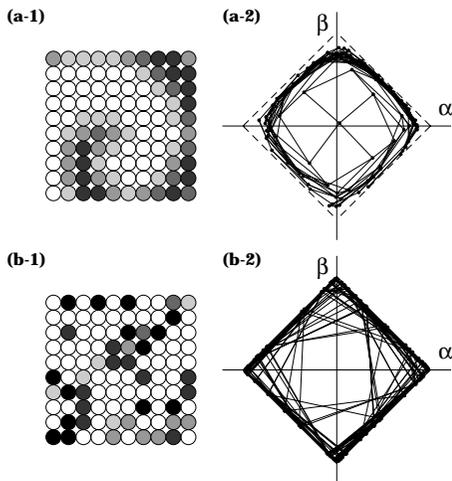


Fig. 4. Snapshots of $x_1^{(u,v)}$ distributions in (a-1) and (b-1), and corresponding phase differences in the α - β plane in (a-2) and (b-2). In (a-2) and (b-2), dots indicate the projected orbitpoints, and the couplings between them are represented by lines. The broken line in (a-2) represents the projected heteroclinic cycle. $D = 10^{-2}$ in system described by (a-1) and (a-2), and a rotating spiral pattern is seen in (a-1). $D = 10^{-4}$ in the system described by (b-1) and (b-2).

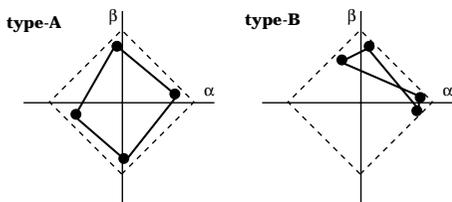


Fig. 5. Typical configurations of a set of four oscillators placed in a unit square.

To understand the structures of the disordered patterns, we note the local phase differences among oscillators. One can project the phase space of each replicator system onto a 2-dimensional plane (the α - β plane):

$$\begin{aligned}\alpha^{(u,v)} &= x_1^{(u,v)} - x_3^{(u,v)}, \\ \beta^{(u,v)} &= x_2^{(u,v)} - x_4^{(u,v)}.\end{aligned}\quad (5)$$

The vertices are projected onto $(\alpha, \beta) = (1, 0), (0, 1), (-1, 0), (0, -1)$ respectively. We plot orbitpoints $[x^{(u,v)}(t)]$ of all oscillators in the α - β plane and connect coupled oscillators with lines.

Typical instantaneous configurations after systems approach sufficiently close to attractors are shown in Fig. 4. Comparing with Fig. 4(a) with a large D , Fig. 4(b) reveals two features as local laws among oscillators. First, all orbitpoints remain close to the heteroclinic cycle. Second, the distances between coupling lines and the origin of the α - β plane are not smaller than a certain positive value. They are observed in every time step of the numerical integrations. Therefore, the relation between the locations of orbitpoints for any two neighboring oscillators never becomes anti-phase. Considering these features, we conclude that the orbitpoint arrangement for every set of the four oscillators that forms unit square of the lattice (represented by dual-lattice point) is classified into 2 types: arrangements

with orbitpoints of four oscillators surrounding the center (type-A) and arrangement for which this is not the case (type-B). Figure 5 displays configurations of these types.

To investigate the difference between the types of arrangements in detail, let us consider a system of 2×2 oscillators as the simplest case. This system is regarded as a system of four oscillators on a 1-dimensional periodic array. Therefore, the situation of type-A corresponds to a traveling wave solution. This is an attractor when the diffusion constant is smaller than about 0.01. Because the phases of oscillators differ, the diffusion effect prevents the orbitpoints from approaching the heteroclinic cycle, and the system retains a uniform frequency. In contrast, if the system is arranged as type-B initially, oscillators are synchronized with each other, and the uniform oscillation approaches the heteroclinic cycle. Therefore, this system needs to be arranged as type-A in order for it to exhibit the oscillation with a uniform frequency.

Again considering a large system, we note only on type-A arrangements, which generate oscillation with a uniform frequency. In such a system, the type-A arrangements distributed on dual-lattice points can be regarded as another type of vortex, whose properties are different from those of the spiral pattern depicted in Fig. 4(a). Therefore, the disordered patterns are understood as representing the coexistence of many vortices, and the distributions of vortices on dual-lattice points are important in the characterization of the patterns.

To make our characterization of the patterns clearer and more systematic, we note another property of this system. As mentioned above, the diffusion effect prevents orbitpoints from approaching a heteroclinic cycle. Thus, a smaller diffusion constant would allow orbitpoints to approach closer to a heteroclinic cycle and remain longer in neighborhoods of fixed points. Therefore, the time during which a orbitpoint moves along with a heteroclinic orbit becomes quite small compared with the time in staying in the neighborhoods of fixed points, and we see all orbitpoints existing in neighborhoods of fixed points in typical snapshots. With such a small diffusion constant, we can systematically generate all possible snapshots as initial conditions by choosing all orbitpoints of replicators in neighborhoods of one of the four fixed points.

Now, we choose the procedure for the algorithmic generation of possible patterns as follows: (i) distribute vortices (type-A arrangements) on the dual-lattice points; (ii) select an initial arrangement of the orbitpoints of oscillators from neighborhoods of the four fixed points to satisfy condition (i). We checked all possible patterns for 3×3 and 4×4 oscillators systems numerically and found that every pattern that satisfies condition (i) has at least one corresponding attractor, and most of them have only one attractor. This means that almost all patterns are uniquely characterized by the distributions of vortices. Therefore, we can estimate the number of attracting patterns with condition (i). Figure 6 displays estimates of this number that were algorithmically determined by computer. Since the condition (i) represents a combinatorial problem the number of patterns grows exponentially as a function of the number of oscillators.

In this paper, we have reported spatially disordered oscillating patterns in coupled heteroclinic cycles. They have been shown to be periodic in time and not

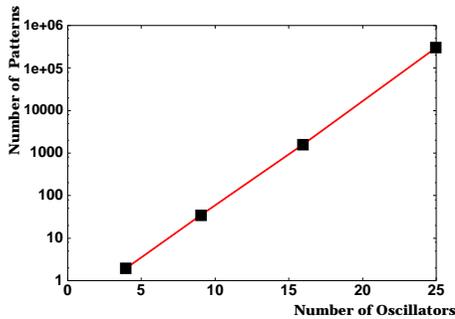


Fig. 6. Estimates of the number of disordered patterns algorithmically calculated with condition (i). The cases of regular squares with 2×2 , 3×3 , 4×4 and 5×5 sites are plotted.

coexistence of vortices whose structure is different from the well-known spiral pattern. This type of disordered oscillating pattern has not been found in any other coupled oscillators. However, other types of spatially disordered stable patterns have been reported in some classes of dynamical systems.^{13)–16)} The relations among these patterns and ours will be discussed in Ref. 12).

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