

Slow Switching near a Blowout Bifurcation

— Yet Another Mechanism —

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(Received September 25, 2002)

An example of slow switching behavior that occurs near a blowout bifurcation is presented. Long lasting chaotic states near an invariant manifold are observed, and their lifetime distribution is obtained for various parameter values. These long lasting states result from a mechanism closely related to a periodic window, in addition to the typical on-off intermittency mechanism.

In many non-equilibrium systems, there are observed *quasi-stable states* that exist for long durations but are not completely stable. Such states with chaotic properties are observed ubiquitously, especially in complex systems. From the viewpoint of dynamical systems, there are several known mechanisms through which there appears such types of nearly stable chaotic behavior: e.g., trapping by ruins of an attractor just broken through a crisis bifurcation (crisis induced intermittency),¹⁾ on-off/in-out intermittency that accompanies a blowout bifurcation in systems with an invariant manifold, which is typically due to symmetry.²⁾⁻⁵⁾

In this paper, we report another mechanism that is capable of producing *quasi-stable chaotic states* in systems near a blowout bifurcation. Though it is closely related to on-off/in-out intermittency, its characteristics are different.

We consider a 2-dimensional map with 1-dimensional invariant manifold as a working model. The map is given by $F : R^2 \rightarrow R^2$, which is defined as follows:**)

$$F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix} = \begin{pmatrix} \kappa x(x^2 - 1) + \varepsilon x y^2 \\ y \exp(-\alpha(x^2 + y^2 - \nu)) \end{pmatrix}, \quad \kappa, \alpha, \varepsilon, \nu \in R.$$

We consider cases in which $\kappa, \alpha, \varepsilon, \nu \geq 0$ and $y \geq 0$.

F leaves $N : \{y = 0\}$ invariant, and the restriction of F on N is given by

$$F|_N \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} f(x; \kappa) \\ 0 \end{pmatrix},$$

where

$$f(x; \kappa) = \kappa x(x^2 - 1).$$

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***) F_2 can be rewritten in a form similar to the appearing in Ref. 4),
 $F_2(x, y) = y \exp(-\alpha(x^2 + y^2 - \nu)) = \tilde{\nu} y \exp(-\alpha(x^2 + y^2))$, with $\tilde{\nu} = \nu^\alpha$.

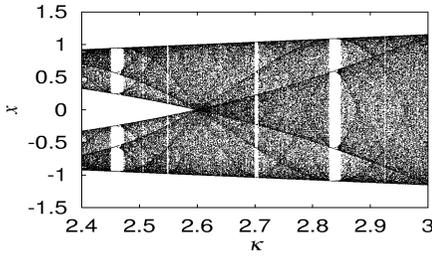


Fig. 1. Bifurcation diagram of $f(x; \kappa)$. $x_n (n = 1000 - 1100)$ is plotted for each κ . κ is changed in steps of 0.0005, and x_0 is set to 0.5.

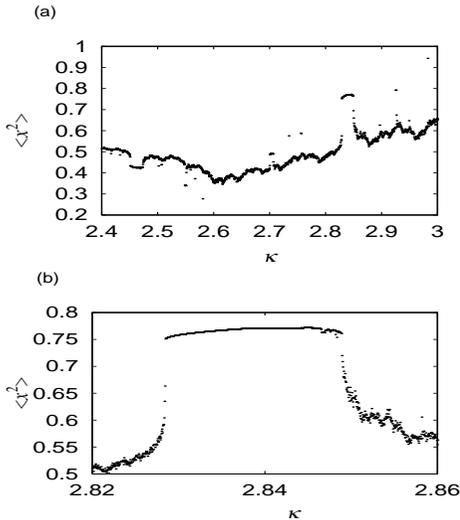


Fig. 2. $\langle x^2 \rangle$ on the attractor of f . The average of $x_n^2 (n = 10000 - 11000)$ is plotted as a function of κ . Here, x_0 is set to 0.5, and the step size of κ is 0.0005 in (a) and 0.00005 in the detailed view of the 4-periodic window in (b).

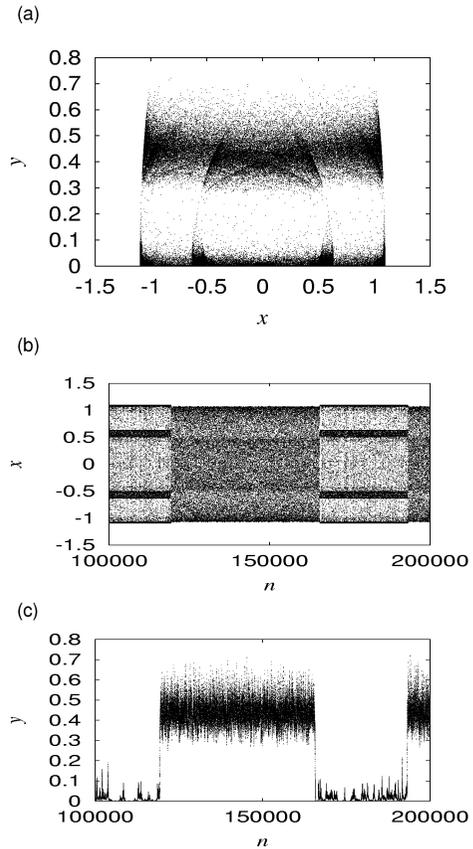


Fig. 3. (a) An orbit of F . (x_n, y_n) is plotted over $n=1 \times 10^6 - 2 \times 10^6$. Here, $\alpha=0.18$, $\nu=\log(1.22)/0.3$ and $(x_0, y_0)=(0.2, e^{-0.69})$. (b) and (c) Temporal sequence of x and y component respectively.

The role of the parameters κ, ν, α and ε can be classified as follows.

- κ affects the dynamics on N .
- ν affects the linear stability of invariant sets on N in the transverse direction.
- α and ε appear in higher-order terms of y . (These two parameters respectively affect speed and scale of the dynamics of the y component.)

The behavior of the trajectories on the invariant manifold N depend solely on κ (Fig. 1), and we studied the κ dependence of $\langle x^2 \rangle$ on the attractor, obtaining the results displayed in Fig. 2. We set κ as $\kappa = 2.8495$, which is just greater than the value corresponding to a window of period 4, and fix $\varepsilon = 0.3$. A blowout bifurcation occurs at $\nu = \nu_{\text{SBR}} = \langle x^2 \rangle_{\text{SBR}} \cong 0.654 \dots$, where the average growth rate of the y component on the SBR measure of $F|_N$ changes sign.

Figure 3 displays an example of the slow switching behavior in question that appears just after the blowout bifurcation.

Let us begin by giving a qualitative sketch of the behavior. Figure 3 indicates that two types of behavior are observed, one after the other, and from Fig. 3(a) it seems natural to divide the phase space roughly into three parts, according to the value of y as follows:

area l ($y < 0.07$) : The orbit remains here for a relatively long duration, behaving chaotically. Even if the orbit escapes from this area, it typically returns after a short duration before reaching the area u . The influence of the ruin of the period 4 window is apparent in the temporal sequence of the x component. Although the period 4 orbit on the invariant manifold N is still stable in the normal direction, the orbit tends to leave N , because $\nu > \nu_{\text{SBR}}$.

area m ($0.07 < y < 0.27$) : The durations of stays in this area are much shorter than stays in the other two areas. Almost periodic behavior of the x component is apparent in this area. The average value of x^2 on the nearly period 4 orbits is relatively large, and thus the growth rate of the y component is negative, while orbits are trapped by this nearly periodic behavior.

area u ($0.27 < y$) : The orbit remains here for relatively long duration, behaving chaotically.

In the case that the change of the y component is sufficiently slow, it would be reasonable to expect that the typical behavior of the x component becomes similar to the orbit of $x = f(x; \kappa')$ with κ' somewhat smaller than κ . The difference of κ' and κ is determined by y (and ε). Thus when y is sufficiently large (as to be in area m), period 4 behavior becomes stable in the x direction, and this period 4 behavior disappears when the y component becomes larger (reaching area u).

Although the orbit in area l enters area m from time to time, due to the nature of area m , it is highly probable that the orbit would be caught by a *stable periodic orbit* and then be pulled back to area l again. There is, however, a small probability that the orbit will reach area u before being caught by the *periodic orbit*.

On the other hand, the behavior of the x component in area u is somewhat similar to the orbit of the map $f(x, \kappa')$ with κ' smaller than the value for a period 4 window. Thus, the average value of x^2 tends to be small and the growth rate of y tends to be positive in the lower part of this area, even if the effect of the nonlinear suppression terms is taken into account. Therefore, when the change of y is slow, i.e. when α is small, it is difficult to enter area m through this part.

We now turn to the statistical properties of this slow switching behavior. For the sake of convenience, we consider a threshold value p in the midst of area m and define two new areas as

$$U = \{(x, y) : p \leq y\},$$

$$L = \{(x, y) : y < p\}.$$

We then obtained the distribution of the duration of the orbit's stay in these two areas, $P(n)$, numerically.

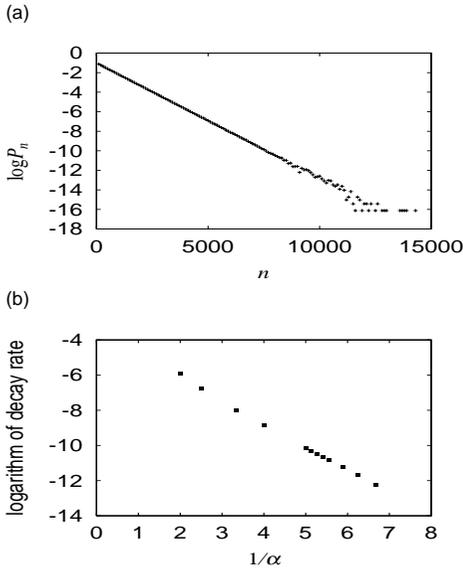


Fig. 4. (a) $\log P(n; \alpha)$ in area U with $\alpha = 0.4$. (b) Logarithm of the decay rate of $P(n; \alpha)$ as a function of $1/\alpha$. $\nu = \log(1.221)/0.3$ in both (a) and (b).

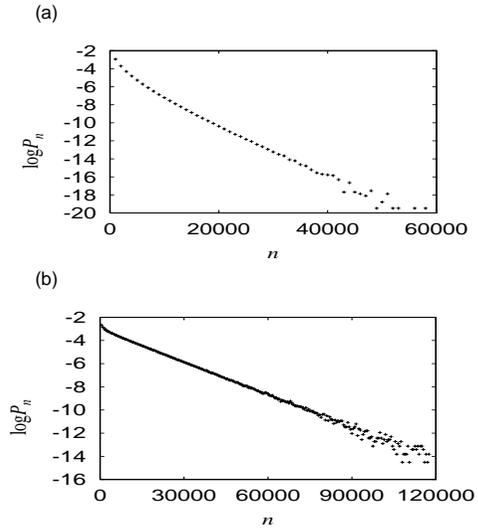


Fig. 5. $\log P(n; \alpha)$ in area L . (a) With a relatively large α ($\alpha = 0.4$), and (b) with a smaller α ($\alpha = 0.18$). $\nu = \log(1.221)/0.3$ in both (a) and (b).

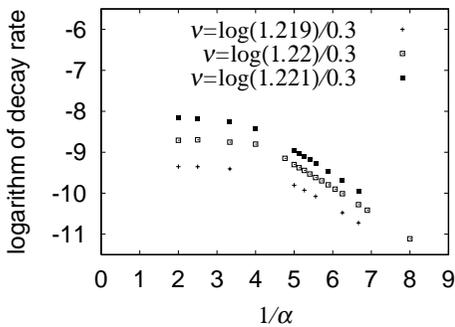


Fig. 6. Logarithm of the decay rate of $\log P(n; \alpha)$ in area L as a function of $1/\alpha$ with $\nu = \log(1.219)/0.3$, $\log(1.22)/0.3$ and $\log(1.221)/0.3$.

Next, we consider the area L . The distribution function $P(n)$ is plotted in Fig. 5. In this case, an apparent bend is observed in the region of relatively small n . This bend is more pronounced for larger α . Contrastingly, the graph tends to lie along a straight line for large n ; i.e., the tail of $P(n)$ seems to decay exponentially for sufficiently large n . The logarithm of the decay rate is plotted again as a function of $1/\alpha$ in Fig. 6. It is found to lie along a straight line for small α and seems to asymptotically approach a constant value for large α . The result of the same calculation for different values of ν is also plotted in this figure, and it indicates that the “saturation level” of the decay rate depends on ν .

For area U , the result (Fig. 4) indicates that the graph of $\log P(n)$ as a function of n is nearly a straight line, except on a very small interval of n values; i.e., the distribution has a tail with exponential decay. Then, we examined the dependence of the decay rate on the parameter α (see Fig. 4). The result lies nearly on a straight line when α is sufficiently small. Thus, for sufficiently small α and large n , $P(n; \alpha)$ can be written

$$\log P(n; \alpha) \sim e^{-\frac{c}{\alpha}n} + o(n).$$

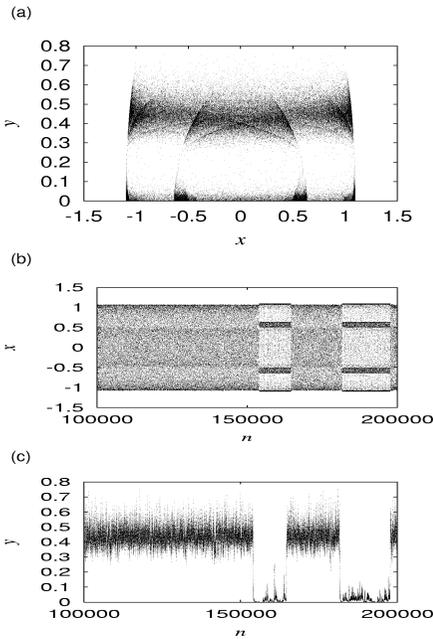


Fig. 7. Orbit of the perturbed system. (a) (x_n, y_n) for $n = 1 \times 10^6 - 2 \times 10^6$. Here, $\alpha = 0.18$, $\nu = \log(1.22)/0.3$, $\delta = 10^{-7}$ and $(x_0, y_0) = (0.2, \exp(-0.69))$. (b),(c) Temporal sequences of the x and y components, respectively.

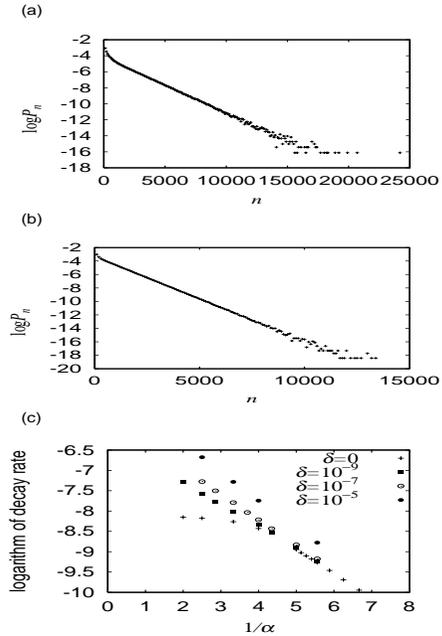


Fig. 8. $\log P(n; \alpha)$ in area L with $\nu = \log(1.221)/0.3$, $\alpha = 0.4$ and $\delta = 10^{-7}, 10^{-5}$ in (a) and (b), respectively. (c) The logarithm of the decay rate of $P(n; \alpha)$ as a function of $1/\alpha$ for $\delta = 10^{-9}$, $\delta = 10^{-7}$ and $\delta = 10^{-5}$. The estimated numerical error is smaller than the size of the symbols.

Thus, for sufficiently large n , $P(n)$ can be written

$$\log P(n; \alpha) \sim e^{-\frac{c}{\alpha}} n + o(n),$$

for sufficiently small α , and

$$\log P(n; \alpha) \sim -sn + o(n),$$

for sufficiently large α , where s and c depend on ν but not on α .

Now we examine how $P(n)$ in area L changes when the invariant manifold N is destroyed by perturbation. As the perturbation, we simply add a small constant δ to $F_2(x, y)$ as

$$F_2(x, y) = y \exp(-\alpha(x^2 + y^2 - \nu)) + \delta.$$

As shown in Fig. 7, the apparent behavior of the orbit changes little as a result of this perturbation: It still goes back and forth slowly between the lower and upper areas. We obtained $P(n)$ for the stay in the lower area and decay rate of its exponential tail for several values of δ . These are compared in Fig. 8. The bend in the graph of n vs $\log P(n)$ is still observed for small δ . However, the length of the interval over which this bend appears decreases as δ increases. Figure 8(c) indicates that the saturation of the decay rate disappears even with a quite small perturbation ($\delta = 10^{-9}$).

From the above results, it is natural to consider there to exist two different factors that cause long stays in the area L . One is the typical on-off intermittency mechanism, which causes a long tail in the distribution of the duration of the laminar phase as a consequence of the random walk of $\log y$ near the invariant manifold N . The lifetime given by this mechanism diverges as $\nu \rightarrow \nu_{SBR} + 0$, while it depends only weakly on α , and decreases drastically as the perturbation destroys the invariant manifold.

The other factor is the difficulty of passing through the “window” area, where the nearly period 4 behavior of the x component is attractive. If α is small, y changes slowly, and it takes a relatively long time to pass through the “window” area from the lower to the upper area. Thus it is highly probable that the orbit is attracted to the period 4 orbit and then pulled back to the lower area. The decay rate of $P(n)$ is proportional to the product of the average frequency of entering the “window” area and the probability of passing through it. Thus its dependence on α is estimated at $\exp(-C/\alpha)$.

When α is large, the main contribution to the probability of very long stays in lower area seems to come from the long tail induced by on-off intermittency. On the other hand, in the small α region (or the case with a broken invariant manifold), the second factor seems to play the main role. The numerical results with various ν (Fig. 6) and with a perturbation (Fig. 8) support this picture.

To summarize, we have observed slow switching behavior that appears near a blowout bifurcation. Here, the orbit continues to behave chaotically, being held in an area near an invariant manifold for a long duration. With regard to this point, it has some similarity to on-off intermittency. However, it is robust with respect to perturbations destroying the invariant manifold. The area is contained by a ‘window’ area, where an “attractive periodic orbit” prevents escape. A nonlinear interaction between the normal (y) and parallel (x) components is essential for the existence of the window area. The phenomenon in question is similar to in-out intermittency with regard to this point. This mechanism is somewhat similar to crisis-induced intermittency, although the nature of the escape route from the quasi-stable state is different. The orbit must pass through a “bottle neck” in the case of crisis-induced intermittency (which is the case with long stays in the upper area in this example), the possible escape route through the “window” area has rich variety. The difference between these two confinement mechanisms would presumably be apparent in the itinerant behavior exhibited by higher-dimensional chaotic systems with multiple quasi-stable states.

This work was supported by a Grant-in-Aid from the Ministry of Education, Culture, Sports, Science and Technology of Japan (No. 13831007).

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