

## On the Boson Number Operator in the Deformed Boson Scheme

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Concerning the Hamiltonian of a harmonic oscillator, a prescription is proposed for making the original form unchanged even after a  $q$ -deformation. The applicability of the prescription is limited, but it can be applied to various cases which are well known.

An idea of  $q$ -deformation serves us an interesting viewpoint for the understanding of the dynamics of many-body systems described by boson operators. For the  $q$ -deformation, the present authors recently proposed a possible scheme in relation to the generalized boson coherent state.<sup>1)–3)</sup> This scheme is, in some sense, an extension of the idea given by Penson and Solomon.<sup>4)</sup> Before entering the central part of this paper, first, we recapitulate the basic idea of Ref. 1).

We treat a boson system described by one kind of boson  $(\hat{c}, \hat{c}^*)$ . As the deformation of  $(\hat{c}, \hat{c}^*)$ , the operator  $(\hat{\gamma}, \hat{\gamma}^*)$  is introduced by the definition

$$\hat{\gamma} = F(\hat{N})\hat{c}, \quad \hat{\gamma}^* = \hat{c}^*F(\hat{N}). \quad (1)$$

Here,  $F(\hat{N})$  denotes a function of the boson number operator,  $\hat{N}$ , defined by

$$\hat{N} = \hat{c}^*\hat{c}. \quad (2)$$

The function  $F(\hat{N})$  obeys a certain condition which is appropriate in each case under investigation. The present deformation is characterized by the function  $F(\hat{N})$ . The commutation relations  $[\hat{c}, \hat{c}^*]$  and  $[\hat{\gamma}, \hat{\gamma}^*]$  are as follows:

$$[\hat{c}, \hat{c}^*] = 1, \quad (3a)$$

$$[\hat{\gamma}, \hat{\gamma}^*] = (\hat{N} + 1)F(\hat{N})^2 - \hat{N}F(\hat{N} - 1)^2. \quad (3b)$$

Later, we discuss the case in which the relation (3b) does not hold. In following our deformed boson scheme, any operator which is expressed as a function of  $(\hat{c}, \hat{c}^*)$ ,  $\hat{O} = O(\hat{c}, \hat{c}^*)$ , is deformed by replacing  $(\hat{c}, \hat{c}^*)$  with  $(\hat{\gamma}, \hat{\gamma}^*)$ :

$$\hat{O}_q = O(\hat{\gamma}, \hat{\gamma}^*). \quad (4)$$

However, in the case of the Hamiltonian, special consideration is necessary for the replacement (4), that is,  $\hat{O} \rightarrow \hat{O}_q$ .

Usually, the Hamiltonian under investigation,  $\hat{H}$ , consists of two terms:

$$\hat{H} = \hat{H}^{(0)} + \hat{H}^{(1)}, \quad (5)$$

$$\hat{H}^{(0)} = \omega \hat{c}^* \hat{c} , \tag{5a}$$

$$\hat{H}^{(1)} = V(\hat{c}, \hat{c}^*) . \tag{5b}$$

Here,  $\hat{H}^{(0)}$  denotes the Hamiltonian of the harmonic oscillator with frequency  $\omega$ . On the other hand, the term  $\hat{H}^{(1)}$  makes the system under the investigation deviate from the harmonic oscillator. With the aid of the  $q$ -deformation, we are able to obtain various terms which make the system deviate from the harmonic oscillator. The formal application of the form (4) to the Hamiltonian (5) gives us

$$\hat{H}_q = \hat{H}_q^{(0)} + \hat{H}_q^{(1)} , \tag{6}$$

$$\hat{H}_q^{(0)} = \omega \hat{\gamma}^* \hat{\gamma} = \omega \hat{N} F(\hat{N} - 1)^2 , \tag{6a}$$

$$\hat{H}_q^{(1)} = V(\hat{\gamma}, \hat{\gamma}^*) = V(F(\hat{N})\hat{c}, \hat{c}^*F(\hat{N})) . \tag{6b}$$

Certainly, we can expect that  $\hat{H}_q^{(1)}$  makes the system deviate from the harmonic oscillator in various forms and, further, we see that  $\hat{H}_q^{(0)}$  also contains the part which plays the same role as  $\hat{H}_q^{(1)}$ . However, if we insist on the original picture provided by the Hamiltonian,  $\hat{H}_q^{(0)}$  should be of the form of the harmonic oscillator with frequency  $\omega$ . This means that  $\hat{H}_q^{(0)}$  is at most of the form

$$\hat{H}_q^{(0)} = E_0 + \omega \hat{c}^* \hat{c} . \tag{7}$$

Here,  $E_0$  denotes a constant. In order to obtain the form (7) in the case of the  $su(2)$ - and the  $su(1, 1)$ -algebraic models, we needed special device for each case which is shown in Refs. 2) and 3).

The main aim of the present note is to give a prescription which leads us to the form (7) automatically. First, we introduce the following operator:

$$\hat{M} = \hat{c}^* \hat{c} + G(\hat{b}) - G(0) . \tag{8}$$

Here,  $\hat{b}$  is defined as

$$\hat{b} = (1/2) \cdot ([\hat{c}, \hat{c}^*] - 1) . \tag{9}$$

The second and the third terms on the right-hand side of the relation (8),  $G(\hat{b})$  and  $G(0)$ , denote the values of a function  $G(x)$  at  $x = \hat{b}$  and 0, respectively. Since  $\hat{b}$  is a null operator, we can see that the operator  $\hat{M}$  is identically the boson number operator  $\hat{N}$ . Applying the formal replacement (4) to the relation (8) leads us to

$$\hat{M}_q = \hat{\gamma}^* \hat{\gamma} + G(\hat{\beta}) - G(0) . \tag{10}$$

Here,  $\hat{\beta}$  is given by

$$\hat{\beta} = (1/2) \cdot ([\hat{\gamma}, \hat{\gamma}^*] - 1) . \tag{11}$$

As can be seen in the relation (3b),  $\hat{\beta}$  is not a null operator, i.e.,  $G(\hat{\beta}) \neq G(0)$ . Substituting the relation (3b) into the definition (11),  $\hat{\beta}$  is given as

$$\hat{\beta} = (1/2) \cdot [(\hat{N} + 1)F(\hat{N})^2 - \hat{N}F(\hat{N} - 1)^2 - 1] . \tag{11a}$$

If the relation (11a) can be solved inversely,  $\hat{N}$  can be expressed in terms of a function of  $\hat{\beta}$ , and we denote it as

$$\hat{N} = g(\hat{\beta}) . \tag{12}$$

As is clear from the form (10),  $\hat{M}_q$  depends on the function  $G(x)$ . In order to fix the form of  $\hat{M}_q$  which is consistent with the form (7), we require the following condition:

$$\hat{\gamma}^* \hat{\gamma} + G(\hat{\beta}) (= \hat{N}F(\hat{N} - 1)^2 + G(\hat{\beta})) = \hat{N} . \tag{13}$$

Then,  $\hat{M}_q$  can be expressed as

$$\hat{M}_q = \hat{c}^* \hat{c} - G(0) . \tag{14}$$

Then, the quantity  $E_0$  in the relation (7) is given by

$$E_0 = -\omega G(0) . \tag{15}$$

The relations (12) and (13) determine the form of  $G(\hat{\beta})$ :

$$G(\hat{\beta}) = \hat{N} - \hat{N}F(\hat{N} - 1)^2 = g(\hat{\beta})[1 - F(g(\hat{\beta}) - 1)^2] , \tag{16}$$

$$G(0) = g(0)[1 - F(g(0) - 1)^2] . \tag{16a}$$

Thus,  $\hat{M}$  and  $\hat{M}_q$  can be expressed in the forms

$$\hat{M} = \hat{c}^* \hat{c} + g(\hat{b})[1 - F(g(\hat{b}) - 1)^2] - g(0)[1 - F(g(0) - 1)^2] , \tag{17}$$

$$\hat{M}_q = \hat{c}^* \hat{c} - g(0)[1 - F(g(0) - 1)^2] . \tag{18}$$

Let us show two examples. Example (i) is the case in which  $F(\hat{N}) = \sqrt{1 \mp \hat{N}/n_0}$ . The upper and the lower signs correspond to the deformations to the  $su(2)$ - and  $su(1, 1)$ -algebras in the Holstein-Primakoff representation, respectively.<sup>2)</sup> In this case, the relation (11a) gives us

$$\hat{\beta} = \mp \hat{N}/n_0 , \quad \text{i.e.,} \quad g(\hat{\beta}) = \mp n_0 \hat{\beta} . \tag{19}$$

Then,  $\hat{M}$  and  $\hat{M}_q$  shown in the forms (17) and (18) can be expressed as

$$\hat{M} = \hat{c}^* \hat{c} + \hat{b}(1 \pm n_0 \hat{b}) , \tag{20}$$

$$\hat{M}_q = \hat{c}^* \hat{c} . \tag{21}$$

Example (ii) is the case related to  $F(\hat{N}) = \sqrt{\hat{N}/n_0 - 1}$ . This case produces the second Holstein-Primakoff representation of the  $su(1, 1)$ -algebra,<sup>2)</sup> and we have

$$\hat{\beta} = \hat{N}/n_0 - 1 , \quad \text{i.e.,} \quad g(\hat{\beta}) = n_0(\hat{\beta} + 1) . \tag{22}$$

Then,  $\hat{M}$  and  $\hat{M}_q$  shown in the forms (17) and (18) can be expressed as

$$\hat{M} = \hat{c}^* \hat{c} + \hat{b}(1 - n_0 \hat{b}) , \tag{23}$$

$$\hat{M}_q = \hat{c}^* \hat{c} - (n_0 + 1) . \tag{24}$$

The results (21) and (24) are natural. In the case of the example (i),  $N$  can take the values  $0, 1, 2, \dots$ , because of the positive-definiteness of  $(1 \mp \hat{N}/n_0)$ . On the other hand, in the case of example (ii),  $N$  can take the values  $(n_0 + 1), (n_0 + 2), (n_0 + 3), \dots$ , because of the positive-definiteness of  $(\hat{N}/n_0 - 1)$ . The results (21) and (24) support these facts.

However, direct formal application of the above procedure is not permitted, for example, for the case

$$F(\hat{N}) = \phi(\hat{N})/\sqrt{\hat{N} + 1} . \tag{25}$$

Here,  $\phi(n)$  is assumed to be a well-behaved function and  $\phi(-1) = 0$ . For example, we have the following form:

$$\phi(n) = \sqrt{[1 - q^{-2(n+1)}]/[1 - q^{-2}]} . \tag{26}$$

This form is a possible modification of the most commonly used form in the  $q$ -deformation,  $\sqrt{[q^{n+1} - q^{-(n+1)}]/[q - q^{-1}]}$ , and the form adopted in Ref. 4),  $q^{n/2}$ .<sup>1)</sup> Here,  $q$  denotes a positive parameter. Formal application of the form (25) to the relation (11a) results in the appearance of the operator  $\hat{N}/\hat{N}$ , which cannot be defined. In this case, the relation (11a) should be rewritten as

$$\begin{aligned} \hat{\beta} &= (1/2) \cdot [(\hat{N} + 1)F(\hat{N})^2 - \hat{Q}_0\hat{N}F(\hat{N} - 1)^2 - 1] \\ &= (1/2) \cdot [\phi(\hat{N})^2 - \hat{Q}_0\phi(\hat{N} - 1)^2 - 1] . \end{aligned} \tag{27}$$

Here,  $\hat{Q}_0$  denotes the projection operator which rejects the vacuum state for the boson  $(\hat{c}, \hat{c}^*)$ . Further, the relation (16) should be rewritten as

$$G(\hat{\beta}) = \hat{N} - \hat{Q}_0\hat{N}F(\hat{N} - 1)^2 = \hat{N} - \hat{Q}_0\phi(\hat{N} - 1)^2 . \tag{28}$$

Since there exists the condition  $\phi(-1) = 0$ , we have the relation  $\hat{Q}_0\phi(\hat{N} - 1)^2 = \phi(\hat{N} - 1)^2$  and, then, the relations (27) and (28) are reduced to the following forms:

$$\hat{\beta} = (1/2) \cdot [\phi(\hat{N})^2 - \phi(\hat{N} - 1)^2 - 1] , \tag{27a}$$

$$G(\hat{\beta}) = \hat{N} - \phi(\hat{N} - 1)^2 . \tag{28a}$$

Using the same procedure as in the case already considered, we can treat the relations (27a), (17) and (18). For the case (26), the following solution is obtained:

$$\hat{\beta} = -(1/2) \cdot (1 - q^{-2\hat{N}}) , \quad \text{i.e.,} \quad g(\hat{\beta}) = -(1/2) \cdot \log(1 + 2\hat{\beta})/\log q , \tag{29}$$

$$\hat{M} = \hat{c}^*\hat{c} + 2\hat{b}/(1 - q^{-2}) - (1/2) \cdot \log(1 + 2\hat{b})/\log q , \tag{30}$$

$$\hat{M}_q = \hat{c}^*\hat{c} . \tag{31}$$

In the case  $\phi(-1) \neq 0$ , a further device should be investigated. In this note, however, we do not address this problem.

We have one more example which should be carefully treated, namely,

$$\phi(n) = 1 . \tag{32}$$

The relation (27) in this case is written as

$$\hat{\beta} = -(1/2) \cdot \hat{Q}_0 . \quad (33)$$

The relation (33) cannot be solved inversely in terms of  $\hat{N} = g(\hat{\beta})$ . Then, we note the relation

$$\lim_{\varepsilon \rightarrow 0} \hat{N}/(\hat{N} + \varepsilon) = \hat{Q}_0 . \quad (34)$$

Replacing  $\hat{Q}_0$  in the relations (27) and (28) with  $\hat{N}/(\hat{N} + \varepsilon)$  and regarding  $\varepsilon$  as an infinitesimal parameter, we formulate the problem. From this procedure, we obtain the following result:

$$\hat{\beta} = -(1/2) \cdot \hat{N}/(\hat{N} + \varepsilon) , \quad \text{i.e.,} \quad g(\hat{\beta}) = -\varepsilon \cdot 2\hat{\beta}/(2\hat{\beta} + 1) , \quad (35)$$

$$\hat{M} = \hat{c}^* \hat{c} + 2\hat{b} - \varepsilon \cdot 2\hat{b}/(2\hat{b} + 1) , \quad (36)$$

$$\hat{M}_q = \hat{c}^* \hat{c} . \quad (37)$$

Here, we have used the relation

$$G(\hat{\beta}) = 2\hat{\beta} - \varepsilon \cdot 2\hat{\beta}/(2\hat{\beta} + 1) , \quad (38)$$

which can be derived through a careful treatment of (28), (12) and (35).

As shown above, we have been able to give a prescription with the aid of which we can arrive at the aim mentioned in the introductory part of this note. Of course, the applicability is limited, however, it can be applied to various known cases. Then, our next interest becomes to investigate the deformation of  $\hat{H}^{(1)}$  which is given in the relation (5b).

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