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# Dirac Sea for Bosons. I

— Formulation of Negative Energy Sea for Bosons<sup>\*)</sup> —

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We propose the formulation of a second quantization of a bosonic theory by generalizing the method of filling the Dirac negative energy sea for the case of fermions. We interpret our results as implying that the correct vacuum for the bosonic theory is obtained by adding minus one boson to each single particle negative energy state while the positive energy states are empty. The boson states are divided into two sectors: the usual positive sector consisting of states with a positive (or zero) number of bosons, and the negative sector, consisting of states with a negative number of bosons. Once a state enters the negative sector, it cannot return to the usual positive sector through an ordinary interaction, due to the presence of a barrier.

To study this problem, a toy model, in which the filling of the empty fermion Dirac sea and the removal of bosons from the negative energy states has not yet been performed, has been proposed. We put forward such a naive vacuum world in the present paper. A subsequent paper<sup>1)</sup> will treat various properties: the analyticity of the wave functions, the interaction and a CPT-like theorem in the naive vacuum world.

### §1. Introduction

There is a well-known method, though not popular nowadays, to carry out the second quantization of relativistic fermions by imagining that there is a priori a so-called naive vacuum in which no fermions, neither positive energy nor negative energy, are present. However, this vacuum is unstable, and for this reason, the negative energy state becomes filled. In this way, the Dirac sea is formed.<sup>3), \*\*\*)</sup> This process by which an initially empty Dirac sea becomes filled seems to make sense only for fermions, to which the Pauli principle applies. In this way the "correct vacuum" is formed out of a "naive vacuum", the former yielding the proper phenomenology. Formally, by filling the Dirac sea, we define the creation operators  $b^+(\vec{p}, s, \omega)$  for holes, which are equivalent to the destruction operators  $a(-\vec{p}, -s, -\omega)$  for negative energies  $-\omega$  and prossessing opposite quantum numbers. This formal rewriting can also be used for bosons, but we have never heard of the filling of negative energy states in this case.

The truly new content as well as the main motivation of the present paper

<sup>&</sup>lt;sup>\*)</sup> This paper is the first part of the revised version of Ref. 2).

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<sup>\*\*\*)</sup> See, for example, Ref. 4) for a historical account.

is to present an idea about how the second quantized field theory in the <u>boson</u> case is analogous to the fermion system before the Dirac sea is filled. However, this bosonic theory analogous to the empty Dirac sea for fermions has the serious drawbacks. First, it has an indefinite "Hilbert space" as its Fock space. Furthermore, the spectrum of its Hamiltonian is bottomless. On the other hand, it has much nicer features than the true vacuum theory, in which the negative energy states are completely filled, namely, the existence of position eigenstates and a description in terms of finite-dimensional wave functions.

At the very end, when the true vacuum for the case of bosons is realized according to the method presented in this paper, we obtain a theory which is exactly the same as the usual one. This leads us to conclude that our approach is valid, but the true vacuum theory itself may not provide new results.

However, "the naively quantized theory", which is an analog of the unfilled Dirac sea for fermions, is theoretically appealing, because it turns out to be a world in which the state of a few particles can be described by wave functions of the positions of these few particles. Remarkably, in contrast to usual relativistic theories, there exist position eigenstates of the particles in the "naive vaccum world". These consist of superpositions of positive and negative energy eigenstates.

The problem of the passage from the naive vacuum world to the usual theory involves, as mentioned above, the addition of "minus one boson" to each negative energy state. In §2, we explicitly formulate the idea of a negative number of bosons mathematically by treating the harmonic oscillator, which is brought into correspondence with a single particle state through the usual second quantization. We extend the spectrum with excitation numbers  $n = 0, 1, 2, \cdots$  to that including all negative integer values,  $n = -1, -2, \cdots$ . This extension can be performed by requiring that the wave function  $\psi(x)$  be analytic in the entire complex x plane, except for an essential singularity at  $x = \infty$ . This requirement replaces the usual condition on the norm of the finite Hilbert space  $\int_{-\infty}^{\infty} |\psi(x)|^2 dx < \infty$ . The conclusion of this study is that the harmonic oscillator has the following two sectors: 1) the usual positive sector consisting of those states with a non-negative number of particles, and 2) the negative sector has an indefinite Hilbert product.

We would like to stress that there is a barrier between the usual positive sector and the negative sector. Due to the presence of this barrier, it is impossible to pass from one sector to the other with usual polynomial interactions. This is due to the existence of an effect that is in some sense an extrapolation of the well-known laser effect. This effect makes it easy to fill an already highly filled single particle state for bosons. This laser effect may vanish when an interaction causes a number of particles to attempt to pass through the barrier. This may offer an explanation of how the barrier prevents us from observing a negative number of bosons.

It may be possible to use as a toy model a formal world in which the usual Dirac sea of fermions is not yet filled and the one boson removal from the negative energy state has not yet been carried out. We study such a toy model, referred to as the naive vacuum model. Specifically, we obtain a theorem analogous to the CPT theorem,<sup>\*)</sup> since the naive vacuum is CPT invariant for neither fermions nor bosons. At first, it might be thought that a strong reflection without the associated inversion of operator order might be sufficient. However, it turns out that this yields the unwanted feature that the sign of the interaction energy is unchanged. The change of the sign is necessary, because under strong reflection, the signs of all energies should be switched. To overcome this problem, we propose a CPT-like symmetry for the naive vacuum world to include a certain analytic continuation. This symmetry is constructed by applying a certain analytic continuation around branch points which appear in the wave function for each pair of particles. It is assumed that we can restrict our attention to a family of wave functions with sufficiently good physical properties. An argument for the validity of this assumption and a proof of a CPT-like theorem are deferred to a subsequent paper.<sup>1</sup>

We present a physical picture that may be of value in developing an intuition for the naive vacuum world. In fact, investigation of the naive vacuum world may be very interesting, because the physics there consists of quantum mechanics of a finite number of particles. Furthermore, the theory is piecewise free in the sense that relativistic interactions become of infinitely short range. Thus, the support for non-zero interactions is the null set, and may be said that the theory is free almost everywhere. However, the very local interactions are communicated only via boundary conditions where two or more particles meet. This makes the naive vacuum world a theoretical toy model. However, it suffers from the following severe drawbacks from the physical point of view:

- The Hamiltonian has no lowest energy state.
- There are states for which the norm-square is negative.
- There are pairs of particles with tachyon-like centers of mass.
- It is natural to work with "anti-bound states" rather than bound states in the negative energy regime.

What we really want to present in the present article is a more dramatic formulation of the relativistic second quantization of boson theory, and this may be thought of as a quantization procedure. Below we formulate the shift of the vacuum for bosons as a shift of boundary conditions in the wave functional formulation of the second quantized theory.

Now, it is interesting to consider whether using the understanding of the second quantization of particles that we describe it could get a better understanding of how to realize a second quantization strings. This was the original motivation of the present work. In the first attempt to construct string field theory by Kaku and Kikkawa,<sup>6)</sup> an infinite momentum frame was used. To us, this appears to be an attempt to avoid the problem of the negative energy states. But this is the root of the trouble to be resolved through the modification of the vacuum described above. Thus our hope would be that by better understanding these Dirac sea problems in our way, it might be possible to formulate new types of string field theories, in which the infinite momentum frame is not necessary.

The present paper is organized as follows. Before presenting a description of how

<sup>&</sup>lt;sup>\*)</sup> The CPT theorem is explained well in Ref. 5).

to quantize bosons in our formulation, we formally consider the harmonic oscillator in §2. This is naturally extended to describe a single particle state that can also have a negative number of particles. In §3, application to even spin particles is described. There, the negative norm-square problems are resolved. In §4, we construct a wave functional formulation of our method. There we explain the change of the convergence and the finite norm conditions. In §5 we illustrate the main point of the formulation of the wave functional by considering a double harmonic oscillator. This is much like a 0 + 1 dimensional world, instead of the usual 3 + 1 dimensional world. In §6, we study the naive vacuum world. Finally, in §7, we give conclusions.

### §2. Analytic harmonic oscillator

In this section we consider as an exercise the formal problem of a harmonic oscillator with the requirement of the analyticity of the wave function. This exercise is crucial for formulating a method to treat bosons with a Dirac sea analogous to that for fermions. In this exercise, the usual requirement that the wave function  $\psi(x)$  be square integrable,

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx < \infty, \tag{2.1}$$

is replaced by the requirement that  $\psi(x)$  be analytic in C, with an essential singularity at  $x = \infty$  allowed. In fact, for this harmonic oscillator we prove the following theorem:

Theorem

1) The eigenvalue spectrum E for the equation

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2\right)\psi(x) = E\psi(x)$$
(2.2)

is given by

$$E = \left(n + \frac{1}{2}\right)\hbar\omega \ (n\varepsilon Z),\tag{2.3}$$

for any integer n.

2) The wave functions for  $n = 0, 1, 2, \cdots$  are the usual ones,

$$\varphi_n(x) = A_n e^{-\frac{1}{2}(\beta x)^2} H_n(\beta x).$$
(2.4)

Here,  $\beta^2 = \frac{m\omega}{\hbar}$ ,  $A_n = \sqrt{\frac{\beta}{\pi^{\frac{1}{2}}2^n n!}}$ , and the functions  $H_n(\beta x)$  are the Hermite polynomials in  $\beta x$ . For  $n = -1, -2, \cdots$  the eigenfunction is given by

$$\varphi_n(x) = \varphi_{-n-1}(ix) = A_{-n-1}e^{\frac{1}{2}(\beta x)^2}H_{-n-1}(i\beta x).$$
(2.5)

3) The inner product is defined as the natural one, given by

$$\langle \psi_1 | \psi_2 \rangle = \int_{\Gamma} \psi_1(x^*)^* \psi_2(x) dx, \qquad (2.6)$$

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where the contour denoted by  $\Gamma$  is taken to be that along the real axis from  $x = -\infty$  to  $x = \infty$ . This contour should be chosen so that the integrand approaches 0 as  $x \to \infty$ , but there remains some ambiguity in this choice of  $\Gamma$ . However, if one chooses the same  $\Gamma$  for all negative n states, the signs of the norm-squares of these states are alternating. In fact, for the path  $\Gamma$  along the imaginary axis from  $-i\infty$  to  $i\infty$ , we obtain

$$\langle \varphi_n | \varphi_m \rangle = \int_{-i\infty}^{i\infty} \varphi_n(x^*)^* \varphi_n(x) dx$$
  
= -(-1)<sup>n</sup>. (2.7)

The proof of this theorem is rather trivial. We may start with consideration of the behavior of a solution to the eigenvalue equation for large x. We assume a wave function of the form

$$\psi(x) = f(x)e^{\pm \frac{1}{2}(\beta x)^2},$$
(2.8)

and we rewrite the eigenvalue equation  $(2\cdot3)$  as

$$\frac{f''(x)}{\beta^2 f(x)} \pm \frac{2f'(x)}{\beta f(x)} \beta x = -\frac{E \mp \frac{1}{2}\omega\hbar}{\omega\hbar}.$$
(2.9)

If we use the approximation that the term  $f''(x)/\beta^2 f(x)$  is dominated by the term  $\pm \frac{2f'(x)}{\beta f(x)}\beta x$  for large |x|, Eq. (2.9) reads

$$\frac{d\log f(x)}{d\log x} = \frac{\mp E + \frac{1}{2}\omega\hbar}{\omega\hbar}.$$
(2.10)

Here, the right-hand side is a constant n, which is yet to be shown to be an integer, and we obtain as the large x behavior

$$f(x) \sim x^n. \tag{2.11}$$

The reason that n must be an integer is that otherwise the function  $x^n$  will have a cut. Thus, requiring that f(x) be analytic except at x = 0, we must have

$$\mp E = -\frac{1}{2}\omega\hbar + n\hbar\omega. \tag{2.12}$$

For the upper sign, the replacement  $n \rightarrow -n - 1$  is made, and we can always write

$$E = \frac{1}{2}\hbar\omega + n\hbar\omega, \qquad (2.13)$$

where n can take not only the non-negative values  $n = 0, 1, 2, \dots$ , but also the negative values  $n = -1, -2 \dots$ .

Indeed, it is easily found that for a negative n, the wave function is

$$\varphi_n(x) = \varphi_{-n-1}(ix) = A_{-n-1}e^{\frac{1}{2}(\beta x)^2}H_{-n-1}(i\beta x).$$
(2.14)

Next, we consider the inner product defined by Eq. (2.6). If the integrand  $\psi_1(x^*)^*\psi_2(x)$  goes to zero as  $x \to \pm \infty$ , the contour  $\Gamma$  can be deformed as usual. But when the integrand does not go to zero, we may have to define the inner product by analytic continuation of the wave functions from the usual positive sector ones that satisfy  $\int |\psi(x)|^2 dx < \infty$ . If we choose  $\Gamma$  to be the path along the imaginary axis from  $x = -i\infty$  to  $x = i\infty$ , the inner product takes the form

$$\langle \varphi_n | \varphi_m \rangle = \int_{-i\infty}^{i\infty} \varphi_n(x^*)^* \varphi_m(x) dx = i \int_{-\infty}^{\infty} \varphi_{-n-1} \left( i \left( i\xi \right)^* \right)^* \varphi_{-m-1} \left( i \left( i\xi \right) \right) d\xi,$$
 (2.15)

where x along the imaginary axis is parameterized by  $x = i\xi$ , with real  $\xi$ . From Eq. (2.15), we obtain for negative n and m,

$$\langle \varphi_n | \varphi_m \rangle = -i(-1)^m \delta_{nm}, \qquad (2.16)$$

and we thus have

$$\|\varphi_n\|^2 = -i(-1)^n.$$
 (2.17)

We note that the norm-square has an alternating sign as a function of n when the contour  $\Gamma$  is kept fixed.

The reason why there is a factor of i in Eq. (2.17) can be understood as follows. When taking the complex conjugation in the definition of the inner product (2.6), the complex conjugate of the contour  $\Gamma$  must be used:

$$\langle \psi_1 | \psi_2 \rangle^* = \int_{\Gamma^*} \psi_1(x^*) \psi_2(x)^* dx.$$
 (2.18)

Thus, if  $\Gamma$  is described by  $x = x(\xi)$  as

$$\Gamma = \{x(\xi) | -\infty < \xi < \infty \quad : \quad \xi \in R\},$$
(2.19)

then  $\Gamma^*$  is given by

$$\Gamma^* = \{ x^*(\xi) | -\infty < \xi < \infty \quad , \quad \xi \in R \} \; . \tag{2.20}$$

Thus, we find

$$\langle \psi_1 | \psi_2 \rangle^* = \int_{-\infty <\xi <\infty} \psi_2(x(\xi)^*)^* \psi_1(x(\xi)) \frac{dx(\xi)^*}{dx(\xi)} dx(\xi), \qquad (2.21)$$

which differs from  $\langle \psi_2 | \psi_1 \rangle$  by the factor  $dx(\xi)^*/dx(\xi)$  in the integrand. In the case  $x(\xi) = i\xi$ , we have  $dx(\xi)^*/dx(\xi) = -1$ , and therefore

$$\langle \psi_1 | \psi_2 \rangle^* = -\langle \psi_2 | \psi_1 \rangle \tag{2.22}$$

for the eigenfunctions of the negative sector. From this relation, we set that the norm-square is purely imaginary.

The above-described convention for the inner product is somewhat unnatural, and for this reason, it may be preferable to change the inner product Eq. (2.6) to a new one defined by

$$\langle \psi_1 | \psi_2 \rangle_{\text{new}} = \frac{1}{i} \langle \psi_1 | \psi_2 \rangle, \qquad (2.23)$$

so as to have the usual relation also in the negative sector,

$$\langle \psi_1 | \psi_2 \rangle_{\text{new}}^* = \langle \psi_2 | \psi_1 \rangle_{\text{new}}.$$
 (2.24)

#### §3. Treatment of the Dirac sea for bosons

In this section we make use of the extended harmonic oscillator described in previous section to quantize bosons.

As is well known in the context of non-relativistic theory, a second quantized system of bosons can be described by using an analogy to a system of harmonic oscillators, one for each state in an orthonormal basis for a single particle. The excitation number n of the harmonic oscillator is identified with the number of bosons present in that state in the basis to which the oscillator corresponds. For instance, if we have a system with N bosons, its state is represented by the symmetrized wave function

$$\psi_{\alpha_1\dots\alpha_N} \ (\vec{x_1}, \cdots, \vec{x_N}), \tag{3.1}$$

where the indices  $\alpha_1, \alpha_2, \dots, \alpha_N$  represent the intrinsic quantum numbers such as spin. In an energy and momentum eigenstate,  $k = (\vec{k}, +)$  or  $k = (\vec{k}, -)$ , where the signs + and - denote those of the energy, we may write

$$K_{\text{pos}} = \{ (\vec{k}, +) | \vec{k} \} , \qquad (3.2)$$

$$K_{\text{neg}} = \{ (\vec{k}, -) | \vec{k} \} . \tag{3.3}$$

Note that we have  $K = K_{\text{pos}} \cup K_{\text{neg}}$ . We then expand  $\psi_{\alpha_1...\alpha_N}(\vec{x_1}, \cdots, \vec{x_N})$  in terms of an orthonormal basis of single particle states,  $\{\varphi_{k;\alpha}(\vec{x})\}$ , with  $k \in K$ . This expansion reads

$$\begin{aligned} |\psi\rangle &= \psi_{\alpha_1\dots,\alpha_N}(\vec{x_1},\cdots,\vec{x_N}) \\ &= \sum C_{k_1,\dots,k_N} \frac{1}{N!} \sum_{\rho \in S_N} \\ \varphi_{k_{\rho(1)}\alpha_1}(\vec{x_1})\varphi_{k_{\rho(2)}\alpha_2}(\vec{x_2})\cdots\varphi_{k_{\rho(N)}\alpha_N}(\vec{x_N}). \end{aligned}$$
(3.4)

The corresponding state of the system of harmonic oscillators is given by

$$|\psi\rangle = \sum_{k_1,\dots,k_N} C_{k_1,\dots,k_N} \prod_{k \in K} |n_k\rangle, \qquad (3.5)$$

where  $|n_k\rangle$  represents the state of the k-th harmonic oscillator.

Next, the harmonic oscillator is extended so as to allow the possibility of negative values of the excitation number,  $n_k$ . Such values correspond to the case in which the number of bosons  $n_K$  in the single particle states is negative. In the nonrelativistic case, one can introduce the creation and annihilation operators  $a_k$  and  $a_k^+$ , respectively. In the harmonic oscillator formalism, these are the step operators for the kth harmonic oscillator,

$$a_k^+|n_k\rangle = \sqrt{n_k + 1} |n_k + 1\rangle , \qquad (3.6)$$

$$a_k |n_k\rangle = \sqrt{n_k} |n_k - 1\rangle. \tag{3.7}$$

It is also possible to introduce creation and annihilation operators for arbitrary states  $|\psi\rangle$  as

$$a^{+}(\psi) = \sum_{k \in K} \langle \varphi_k | \psi \rangle a_k^+, \qquad (3.8)$$

$$a(\psi) = \sum_{k \in K} a_k \langle \varphi_k | \psi \rangle, \tag{3.9}$$

where the inner product is defined by  $\int d^3x \varphi^*(x) i \overleftrightarrow{\partial}_0 \psi(x)$ . We then find

$$[a(\psi'), a^{+}(\psi)] = \sum_{k,k'} \langle \psi' | \varphi_{k'} \rangle [a_{k'}, a_{k}] \langle \varphi_{k} | \psi \rangle$$
$$= \langle \psi' | \psi \rangle, \qquad (3.10)$$

in which the right-hand side contains an indefinite Hilbert product. Thus, if we perform this naive second quantization, the possible negative norm-square is inherited by the quantized states in the Fock space.

Suppose that we choose the basis such that for some subset  $K_{\text{pos}}$  the norm square is unity, i.e.,

$$\langle \varphi_k | \varphi_k \rangle = 1 \quad \text{for} \ k \in K_{\text{pos}},$$
 (3.11)

while for the complement set  $K_{\text{neg}} = K \setminus K_{\text{pos}}$  it is -1, i.e.,

$$\langle \varphi_k | \varphi_k \rangle = -1 \quad \text{for } k \in K_{\text{neg.}}$$

$$(3.12)$$

Thus, any component of a Fock space state must have a negative norm-square if it has an odd number of particles in states of  $K_{\text{neg}}$ .

We thus have the following signs of the norm-square in the naive second quantization:

$$\langle n_k | m_k \rangle = \delta_{n_k m_k} (-1)^{n_k}, \qquad (3.13)$$

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for  $k \in K_{\text{neg}}$ , where  $n_k$  and  $m_k$  denote the usual nonzero levels. Using our extended harmonic oscillators, we end up with a system whose values of the norm-squared are as follows:

for  $k \in K_{\text{pos}}$ ,

and for  $k \in K_{\text{neg}}$ ,

We should bear in mind here that the trouble involving the negative norm-square can be solved by putting minus one particle into each state with  $k \in K_{\text{neg}}$ . When this is done, these states are restricted to negative numbers of particles. Thus, we have to use the inner product  $\langle n_k | m_k \rangle = i \delta_{n_k m_k}$ , which makes the Fock space sector a good positive definite Hilbert space, apart from the overall factor of *i*.

We formulate our procedure in the following. The naive vacuum can be constructed from harmonic oscillators states as

$$|\operatorname{naive vac}\rangle = \prod_{k \in K} |0\rangle_{\operatorname{ssc}},$$
 (3.16)

where  $|0\rangle_{kth}_{osc}$  denotes the vacuum state of the *k*th harmonic oscillator. However, the correct vacuum is given by

$$|\operatorname{correct vac}\rangle = \prod_{k \in K_{\text{pos}}} |0\rangle_{\substack{k \text{th} \\ \text{osc}}} \cdot \prod_{k \in K_{\text{neg}}} |-1\rangle_{\substack{k \text{th}, \\ \text{osc}}},$$
 (3.17)

where the states  $|-1\rangle$  in  $K_{\text{neg}}$  are those with minus one particles.

We proceed to the case of relativistic integer spin particles, whose inner product is indefinite by Lorentz invariance,

$$\int \psi^*(\vec{x},t) \stackrel{\leftrightarrow}{\partial_t} \psi(\vec{x},t) d^3 \vec{x} . \qquad (3.18)$$

For the simplest scalar field case, the energy of the naive vacuum is given by

$$E_{\text{naive vac}} = \sum_{k \in K} \frac{1}{2} \omega_k = 0.$$
(3.19)

By adding minus one particle to each negative energy state,  $\varphi_{k-}$ , with  $k \in K_{\text{neg}}$ , the second quantized system is brought into such a sector that it is in the ground state. This state is the correct vacuum. Its energy is given by

$$E_{\text{correct vac}} = \sum_{k \in K} \frac{1}{2} \omega_k - \sum_{k \in K_{\text{neg}}} \frac{1}{2} \omega_k \tag{3.20}$$

$$=\sum_{k\in K}\frac{1}{2}|\omega_k|=\sum_{k\in K_{\text{pos}}}\omega_k.$$
(3.21)

It should be stressed that we obtain the ground state in this way only inside the sector. In fact, with the single particle negative energies for bosons, the total Hamiltonian may have no lowest energy state. For this reason, if we do not add minus one particle to each single particle negative energy state, there may exist a series of states whose energies go to  $-\infty$ . However, by adding minus one particle, we obtain a state of the second quantized system in which there exists a barrier, due to the laser effect. This barrier prevents the system from falling back to lower energies, as long as the interaction is restricted to polynomials in  $a_k^+$  and  $a_k$ .

In the above calculation, for the relativistic case, we have

$$E_{\text{correct vac}} > E_{\text{naive vac}}.$$
 (3.22)

Thus, at first sight, the correct vacuum appears to be unstable. However, the question of which vacuum has a lower energy is not important with regard to the stability of a certain vacuum. Rather, the important thing is the range of allowed energies for the sector of the vacuum under consideration. For this reason, we define the energy range  $E_{\text{range}}$  of the vacuum by

$$E_{\text{range}}(|\text{vac}\rangle) = \{E\},\tag{3.23}$$

where E denotes the energy in a state that can be reached from  $|vac\rangle$  by some operators that are polynomials in  $a^+$  and a. Thus, for the naive vacuum we have

$$E_{\text{range}}(|\text{naive vac}\rangle) = (-\infty, \infty),$$
 (3.24)

while for the correct vacuum we have

$$E_{\text{range}}(|\text{correct vac}\rangle) = \left[\sum_{k \in K} \frac{1}{2} |\omega_K|, \infty\right].$$
 (3.25)

Once the vacuum is brought into the correct vacuum state, it is no longer possible to add particles to the state with  $K_{\text{neg}}$ , due to the presence of the barrier. However, it is possible to subtract particles. Thus  $a_k$  with  $k \in K_{\text{neg}}$  can act on  $|-1\rangle_{k\text{th}}$  an arbitrary number of times as

$$(a_k)^n |-1\rangle_{\text{kth}} = \sqrt{|n|!} |-1-n\rangle_{\text{osc}}^{\text{kth}}.$$
 (3.26)

We can regard these subtractions as holes which correspond to the addition of antiparticles.

It is natural to switch notation from that with a dagger to that without by defining

$$b^+(-\overrightarrow{k}, \operatorname{anti}) = a(\overrightarrow{k}, -),$$
 (3.27)

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and vice versa where  $k = (\vec{k}, -)$  is a  $\omega < 0$  state with the 3-momentum  $\vec{k}$ . The operator  $b^+(-\vec{k}, \text{anti})$  causes the creation of an antiparticle with momentum  $\vec{k}$  and positive energy  $-\omega > 0$ . This is exactly the usual method of treatment for the second quantization of bosons. The commutator of these operators is given by

$$[b(\vec{k}, \text{anti}), b^+(\vec{k'}, \text{anti})] = \delta_{\vec{k}\vec{k'}}.$$
(3.28)

It should be noted that in the boson case, the antiparticles are also holes. Before closing this section, we discuss two important issues. The first is that there are potentially four possible vacua in our approach to the quantization.

We have argued that we can obtain the correct vacuum by modifying the naive vacuum so that one fermion is added and one boson removed from each single particle negative energy state. This presents the possibility of considering a naive vacuum and the associated world of states in which there exist a few extra particles. The naive vacuum should be considered a toy model for the study of the correct vacuum. It should be mentioned that once we start with one of the vacua and proceed by adding particles to the negative energy states or removing particles from them, we may do the same for positive energy states. In this way, we can think of four different vacua, which are illustrated symbolically as types (a)–(d) in Fig. 1.

As an example, let us consider the type (c) vacuum. In this vacuum, the positive energy states are modified by adding one fermion to the positive energy states and removing one boson from them, while the negative energy states are not modified. Therefore, the single particle energy spectrum has a top but no bottom. Experimentally, it would not be possible to distinguish this and the reversed convention for the energy as long as a free system is concerned. However, with the reversed convention, we would have negative norm-squares for all bosons, and the interactions would act in the opposite manner. We show in subsequent sections that there is a trick of the analytic continuation of the wave function that can be used to circumvent this inversion of the interaction.

The second issue regards the CPT operation on the four vacua. The CPT operation on the naive vacuum denoted type (a) in Fig. 1 does not transform into the same vacuum. The reason is that under the charge conjugation operator C, all the holes in the negative energy states are, from the correct vacuum point of view, replaced by the particles of the corresponding positive energy states. Thus, acting with the CPT operator on the naive vacuum, it is transformed into the type (c) vacuum, because the positive energy states are modified, while the negative ones remain the same. This implies that in the naive vacuum, CPT symmetry is spontaneously broken. However, in the subsequent paper "Dirac sea for bosons. II", <sup>1)</sup> we present another CPT-like theorem concerning a situation in which the CPT-like symmetry is preserved in the naive vacuum but broken in the correct one.

Before closing this section, we mention some properties of the world around the naive vacuum, where there are only a few particles. The term "the world around



Fig. 1. Four types of vacua. There are four possible types of vacua for bosons as well as fermions. Here, the vertical axis indicates the energy level. In (a) - (d), the shaded regions represent states that are all filled by one particle for fermions and minus one particle for bosons. The unshaded regions represent empty states.

a vacuum" is used for a Hilbert space with a superposition of such states that it deviates from the vacuum in question by a finite number of particles and that bosons do not cross the barrier. Because the naive vacuum contains no particles, we can add a positive number of particles, each of which can have either a positive or negative energy. The correct vacuum may similarly have a finite number of particles and holes, in addition to the negative energy seas.

# §4. Wave functional formulation

In this section we develop the wave functional formulation of field theory in the naive vacuum world.

In the field theoretical formulation with the naive field quantization

$$\varphi(\vec{x},t) = \sum_{\vec{p},\text{sign}} \frac{1}{\sqrt{|\omega|}} a(\vec{p},\text{sign}) e^{-i\omega t + i\vec{p}\cdot\vec{x}}, \qquad (4.1)$$

$$\pi(\vec{x},t) = \sum_{\vec{p},\text{sign}} \frac{1}{\sqrt{|\omega|}} a(\vec{p},\text{sign}) \cdot (\text{sign}) = e^{-i\omega t + i\vec{p}\cdot\vec{x}}, \qquad (4.2)$$

we have the wave functional  $\Psi[\varphi]$ . For each eigenmode  $\omega \varphi_{\overrightarrow{p}} + i\pi_{\overrightarrow{p}}$ , where  $\varphi_{\overrightarrow{p}}$  is the 3-spatial Fourier transform of  $\varphi(x)$ , and  $\pi_{\overrightarrow{p}}$  is its conjugate momentum, we have an extended harmonic oscillator described in §2. In order to see how to put the naive vacuum world into a wave functional formulation, we investigate the Hamiltonian and the boundary conditions for single particle states with a general norm-square.

Let us imagine that we employ the convention in which the n-particle state is

$$A_n H_n(x), \tag{4.3}$$

with  $H_n$  a Hermite polynomial. Thus we have

$$|n\rangle = A_n H_n(x)|0\rangle. \tag{4.4}$$

On the other hand, the nth excited state for the harmonic oscillator is given by

$$A_n H_n(x) \beta e^{-\frac{1}{2}(\beta x)^2},\tag{4.5}$$

with  $\beta^2 = \frac{\hbar}{m\omega}$ . We can vary the normalization while maintaining the convention

$$\langle n|n\rangle = \beta^{-2n} \langle 0|0\rangle. \tag{4.6}$$

We may consider  $\beta^{-2}$  as the norm square of the single particle state corresponding to the harmonic oscillator.

Now the Hamiltonian of the harmonic oscillator can be expressed in terms of  $\omega$  and  $\beta^{-2}$  as

$$H = -\frac{\omega}{2\langle s.p.|s.p.\rangle} \frac{d^2}{dx^2} + \frac{1}{2} \langle s.p.|s.p.\rangle \omega x^2, \qquad (4.7)$$

where  $|s.p.\rangle$  denotes the single particle state, and thus

$$\langle s.p.|s.p.\rangle = m\omega = \beta^{-2}, \qquad (4.8)$$

with  $\hbar = 1$ . Therefore, we obtain the Hamiltonian

$$H = -\frac{1}{2}\beta^2 \omega \frac{d^2}{dx^2} + \frac{1}{2}\beta^{-2}\omega x^2.$$
 (4.9)

Now, we remark that to make  $\langle s.p.|s.p. \rangle$  negative for negative  $\omega$ ,  $\beta$  must be purely imaginary. Thus,  $e^{-\frac{1}{2}(\beta\chi)^2}$  blows up so that the wave functions become like those in the extended negative sector discussed in the previous sections.

In passing to the correct vacuum world by removing one particle from each negative energy state, the boundary conditions for the wave functional are changed so as to converge along the real axis for all the modes. It should be kept in mind here that the boundary conditions are along the imaginary axis for the negative energy modes in the naive vacuum.

From the fact that the form of the Hamiltonian in the wave functional formalism must be the same as that for the correct vacuum, we can easily write down the Hamiltonian. For instance, using the conjugate variable  $\pi$ ,

$$\pi(\vec{x}) = -i\frac{\delta}{\delta\varphi(\vec{x})},\tag{4.10}$$

the free Hamiltonian becomes

$$H_{\text{free}} = \int \frac{1}{2} \left\{ |\pi(\vec{x})|^2 + |\nabla \varphi(\vec{x})|^2 + m |\varphi(\vec{x})|^2 \right\} d^3 \vec{x} .$$
 (4.11)

This acts on the wave functional as

$$H_{\text{free}}\Psi[\varphi] = \frac{1}{2} \int \left\{ -\frac{\delta^2}{\delta\varphi(\vec{x})^2} + |\nabla\varphi(\vec{x})|^2 + m^2 |\varphi(\vec{x})|^2 \right\} \Psi[\varphi].$$

The inner product for the functional integral is given by

$$\langle \Psi_1 | \Psi_2 
angle = \int \Psi_1 [(\operatorname{Re} \varphi)^*, (\operatorname{Im} \varphi)^*]^*$$
  
 $\cdot \Psi_2 [\operatorname{Re} \varphi, \operatorname{Im} \varphi] \mathcal{D} \operatorname{Re} \varphi \cdot \mathcal{D} \operatorname{Im} \varphi$ 

where the independent functions are  $\operatorname{Re}\varphi(\vec{x})$  and  $\operatorname{Im}\varphi(\vec{x})$ . In order to describe the wave functional theory of the naive vacuum world, we construct a formulation in terms of the convergence condition along the real function space for  $\operatorname{Re}\varphi$  and  $\operatorname{Im}\varphi$ . In fact, we use a representation in which  $\Psi[\operatorname{Re}\varphi, \operatorname{Im}\pi]$  is expressed in terms of  $\varphi$  and  $\pi$ .

We would like our formulation to be such that the boundary conditions for the quantity  $\omega \varphi_{\overrightarrow{k}} + i\pi_{\overrightarrow{k}}$  are convergent on the real axis for  $\omega > 0$  and convergent on the imaginary axis for  $\omega < 0$ . We can consider the real and imaginary parts of  $(\omega \varphi_{\overrightarrow{k}} + i\pi_{\overrightarrow{k}})$  separately. Then, the requirement of convergence in the correct vacuum should be that for  $\omega < 0$  the formal expressions

$$\operatorname{Re}(\omega\varphi_{\vec{k}} + i\pi_{\vec{k}}) = \frac{\omega}{2} \left\{ (\operatorname{Re}\varphi)_{\vec{k}} + (\operatorname{Re}\varphi)_{-\vec{k}} \right\} - \frac{1}{2} \left\{ (\operatorname{Im}\pi)_{\vec{k}} + (\operatorname{Im}\pi)_{-\vec{k}} \right\} (4.12)$$

and

$$\operatorname{Im}(\omega\varphi_{\overrightarrow{k}} + i\pi_{\overrightarrow{k}}) = \frac{\omega}{2} \left\{ (\operatorname{Im}\varphi)_{\overrightarrow{k}} + (\operatorname{Im}\varphi)_{-\overrightarrow{k}} \right\} + \frac{1}{2} \left\{ (\operatorname{Re}\pi)_{\overrightarrow{k}} + (\operatorname{Re}\pi)_{-\overrightarrow{k}} \right\} (4.13)$$

are purely imaginary along the integration path for which the convergence is required.

We can use the following parameterization in terms of the two real functions  $\chi_1$  and  $\chi_2$ :

$$Re\varphi = -(1+i)\chi_1 - (1-i)\chi_2,$$
  
Im $\pi = (1-i)\chi_1 + (1+i)\chi_2.$ 

With this parameterization, the phases of  $\omega \varphi_{\overrightarrow{k}} + i \pi_{\overrightarrow{k}}$  lie in the intervals

$$]-\frac{\pi}{4}$$
,  $\frac{\pi}{4}[$  for  $\omega > 0$ 

and

$$]\frac{\pi}{4} , \frac{3\pi}{4}[$$
 for  $\omega < 0$ ,

modulo  $\pi$ . They provide the boundary conditions for the naive vacuum world when convergence of the  $\mathcal{D}\chi_1\mathcal{D}\chi_2$  integration is required.

In this way, we find the naive vacuum world with the usual wave functional Hamiltonian operator. However, we do not require the usual convergence condition

$$\int \Psi(\operatorname{Re}\varphi, \operatorname{Im}\varphi)^* \Psi(\operatorname{Re}\varphi, \operatorname{Im}\varphi) \mathcal{D}\operatorname{Re}\varphi \mathcal{D}\operatorname{Im}\varphi < \infty, \qquad (4.14)$$

but, instead,

$$\langle \Psi | \Psi \rangle = \int \Psi [(\operatorname{Re}\varphi)^*, (\operatorname{Im}\pi)^*]^* \Psi [\operatorname{Re}\varphi, \operatorname{Im}\varphi] \mathcal{D}\chi_1 \mathcal{D}\chi_2 < \infty, \qquad (4.15)$$

where the left-hand side is defined along the path with  $\chi$ -parameterization. The inner product corresponding to this functional contour is

$$\langle \Psi_1 | \Psi_2 \rangle = \int \Psi_1 [-(1-i)\chi_1 - (1+i)\chi_2, (1+i)\chi_1 + (1-i)\chi_2]^* \Psi_2 [-(1+i)\chi_1 - (1-i)\chi_2, (1-i)\chi_1 + (1+i)\chi_2] \mathcal{D}\chi_1 \mathcal{D}\chi_2.$$
 (4·16)

This quantity is not positive definite, and this is related to the fact that there are many negative norm-square states in the Fock space in the naive vacuum world.

The method of filling the Dirac sea vacuum for fermions is now extended to the case of bosons that in the naive vacuum we have the strange convergence condition Eq. (4.14). We then transform to the correct vacuum by switching the boundary conditions to those with convergence along the real axis, e.g., with  $\text{Re}\varphi$  and  $\text{Im}\pi$  real.

#### §5. Double harmonic oscillator

To illustrate how our functional formalism works, we consider as a simple example a double harmonic oscillator. This example is relevant for the following three reasons:

- 1) It is a subsystem of a field theory that consists of two single particle states with  $p^{\mu} = (\vec{p}, \omega(\vec{p}))$  and  $-p^{\mu} = (-\vec{p}, -\omega(\vec{p}))$  for  $\omega(\vec{p}) > 0$ .
- 2) It could correspond to a single 3-position field, where the gradient interaction is ignored.
- 3) It is a 0 + 1- dimensional field theory model.

We start by describing the spectrum for the free case corresponding to a twostate system in which the two states have opposite values of  $\omega$ 's. The boundary conditions in the naive vacuum world are given by

$$\int \psi \left( (\operatorname{Re}\varphi)^{*}, (\operatorname{Im}\Pi)^{*} \right)^{*} \psi \left( \operatorname{Re}\varphi, \operatorname{Im}\Pi \right) d\chi_{1}\chi_{2}$$
  
= 
$$\int \psi \left( -(1-i)\chi_{1} - (1+i)\chi_{2}, (1+i)\chi_{1} + (1-i)\chi_{2} \right)^{*} \psi \left( -(1+i)\chi_{1} - (1-i)\chi_{2}, (1-i)\chi_{1} + (1+i)\chi_{2} \right) d\chi_{1}d\chi_{2} < \infty, \quad (5\cdot1)$$

which is similar to Eq. (4.16). However, in Eq. (5.1), the quantities  $\chi_1$  and  $\chi_2$  are not functions, but merely real variables. Here, we use a mixed representation in terms of the position variables  $\text{Re}\varphi$  and  $\text{Im}\varphi$  and the conjugate momenta

$$\operatorname{Re}\pi = -i\frac{\partial}{\partial\operatorname{Re}\varphi}, \ \operatorname{Im}\pi = -i\frac{\partial}{\partial\operatorname{Im}\varphi}.$$
 (5.2)

The Hamiltonian is that of a rotationally symmetric two dimensional oscillator, because the two values of  $\omega$ 's are opposite. Then, from Eq. (4.7), the coefficient of  $\frac{\partial^2}{\partial \text{Im}\varphi^2}$  is

$$\frac{-\omega}{2 <= \langle s.p.|s.p.\rangle},\tag{5.3}$$

and that of  $(\mathrm{Im}\varphi)^2$  is

$$\frac{1}{2}\langle s.p.|s.p.\rangle\omega,\tag{5.4}$$

where  $|s.p.\rangle$  denotes the single particle state. These coefficients are the same for both oscillators, and thus the Hamiltonian reads

$$\begin{split} H &= -\frac{1}{2} |\omega| \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi^*} + \frac{1}{2} |\omega| \varphi^* \varphi \\ &= \frac{1}{2} |\omega| \left( -\frac{\partial^2}{\partial \text{Re}\varphi^2} - \frac{\partial^2}{\partial \text{Im}\varphi^2} + \text{Re}\varphi^2 + \text{Im}\varphi^2 \right). \end{split}$$

This is expressed in the mixed representation as

$$H = \frac{1}{2} |\omega| \left( -\frac{\partial^2}{\partial \text{Re}\varphi^2} + \text{Re}\varphi^2 + \text{Im}\pi^2 - \frac{\partial^2}{\partial \text{Im}\pi^2} \right).$$
(5.5)

We can express H in terms of the real parameterization with  $\chi_1$  and  $\chi_2$  by using the relations

$$Re\varphi = -(1+i)\chi_1 - (1-i)\chi_2 ,$$
  
Im $\pi = (1-i)\chi_1 + (1+i)\chi_2.$ 

It is convenient to define

$$\chi_{\pm} = \sqrt{2}(\chi_2 \pm \chi_1), \tag{5.6}$$

so that the Hamiltonian can be simply expressed as

$$H = \frac{1}{2} |\omega| \left( \frac{\partial^2}{\partial \chi_-^2} - \chi_-^2 - \frac{\partial^2}{\partial \chi_+^2} + \chi_+^2 \right).$$
(5.7)

The inner product takes the form

$$\langle \tilde{\psi}_1 | \tilde{\psi}_2 \rangle = \int \tilde{\psi}_1(-\chi_-, \chi_+)^* \tilde{\psi}_2(\chi_-, \chi_+) d\chi_- d\chi_+,$$
(5.8)

where

$$\tilde{\psi}_i(\chi_-,\chi_-) = \psi_i(-\sqrt{2}\chi_+ + i\sqrt{2}\chi_-,\sqrt{2}\chi_+ + i\sqrt{2}\chi_-).$$
(5.9)

As expected, the Hamiltonian turns out to be that of two uncoupled harmonic oscillators expressed in terms of  $\chi_{-}$  and  $\chi_{+}$ . The  $\chi_{+}$  oscillator is the usual one, while  $\chi_{-}$  differs from the usual one in two ways. First, it has an overall negative sign. Second, in the definition of the inner product,  $-\chi_{-}$  is used instead of  $\chi_{-}$  in the bra wave function. This difference is equivalent to removing the parity transformation  $\chi_{-} \rightarrow -\chi_{-}$  from the inner product.

The energy spectrum is made up of all combinations of positive contributions  $|\omega|(n_+ + \frac{1}{2})$  and negative contributions  $-|\omega|(n_- + \frac{1}{2})$ . Therefore we have

$$E = |\omega|(n_{+} - n_{-}). \tag{5.10}$$

The norm-squares of these combinations of eigenstates are  $(-1)^{n-1}$ , which is equal to the parity under the  $\chi_{-}$  parity transformation  $\chi_{-} \rightarrow -\chi_{-}$ .

If we consider a single particle state, the charge (i.e., the number of particles) is given by

$$Q = \frac{i}{4} \{\pi^+, \varphi\} - \frac{i}{4} \{\varphi^+, \pi\}$$
  
=  $\frac{1}{2} \chi_+^2 - \frac{1}{2} \frac{\partial^2}{\partial \chi_+^2} + \frac{1}{2} \chi_-^2 - \frac{1}{2} \frac{\partial^2}{\partial \chi_-^2} - 1.$  (5.11)

This is simply the sum of two harmonic oscillator Hamiltonians with the same unit frequency. Thus, the eigenvalue Q' of Q can take only non-negative integer values.



Fig. 2. Charge vs energy in a two-state system. The energy E versus the charge Q (i.e., the number of particles) in the two-state system described in the main text. This two state system actually represents a massive boson theory in 1 time+0 space dimensions. Here, a dot alone indicates that there is one Fock space state, while a dot within a circle indicates that there are two Fock space states with quantum numbers E and Q, and a dot with "3" indicates that there are three states. The triangles of dots are to be understood as extending to infinity. The naive vacuum is depicted in (a), while the true vacuum is depicted in (b).

For a given value Q', which is the number of particles in either of the two states, the energy can vary from  $E = -|\omega|Q'$  to  $E = |\omega|Q'$  in steps of  $2|\omega|$ . Thus, we have

$$n_{-} = Q', Q' - 1, Q' - 2, \cdots, 0 \tag{5.12}$$

for the negative  $\omega$  states, while for the positive energy states, we have

$$n_{+} = Q - n_{-} . (5.13)$$

Therefore the energy given in Eq. (5.10) can be written as

$$E = |\omega|(Q - 2n_{-}), \tag{5.14}$$

which is illustrated in Fig. 2(a). By charging to the convergence condition along the real axis, we obtain the usual theory with correct vacuum [see Fig. 2(b)].

The wave function of the naive vacuum is given by

$$\psi_{n\nu} = N \exp\left(-\frac{1}{2}\chi_{-}^{2} - \frac{1}{2}\chi_{+}^{2}\right), \qquad (5.15)$$

with normalization constant N. We can transform Eq. (5.14) in the mixed transformation back to the position representation by applying the Fourier transformation

$$\psi_{\mathrm{n}\upsilon}(\mathrm{Re}\varphi,\mathrm{Im}\varphi) = \int e^{i\mathrm{Im}\Pi\cdot\mathrm{Im}\varphi}\psi_{\mathrm{n}\upsilon}(\mathrm{Re}\varphi,\mathrm{Im}\Pi)d\mathrm{Im}\Pi$$

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$$= N \int e^{i \operatorname{Im} \Pi \cdot \operatorname{Im} \varphi} e^{\operatorname{Im} \Pi \cdot \operatorname{Re} \varphi} d \operatorname{Im} \Pi$$
$$= N \delta(\operatorname{Im} \varphi - i \operatorname{Re} \varphi). \tag{5.16}$$

Here, the  $\delta$ -function is regarded as a functional linear in test functions that are analytic and decays faster than any power in real directions and no faster than a certain exponential in imaginary directions. We call this function the distribution class Z', following Gel'fand and Shilov.<sup>7</sup>) Thus our naive vacuum wave function is a  $\delta$ -function that belongs to Z'.

By applying operators consisting of polynomials in creation and annihilation operators to the naive vacuum state we obtain an expression of the form

$$\sum_{m=0,1,\dots} a_{n,m} (\operatorname{Re}\varphi - i\operatorname{Im}\varphi)^n \delta^{(m)} (\operatorname{Re}\varphi + \operatorname{Im}\varphi).$$
(5.17)

Thus, the wave functions of the double harmonic oscillator in the naive vacuum world take the form of Eq. (5.17).

As long as the charge Q is conserved, even with an interaction term such as an anharmonic double oscillator with phase rotation symmetry, only states of the form given in Eq. (5.17) can mix. For such a finite quantum number Q, there is only a finite number of these states of this form. Therefore, even solving the anharmonic oscillator problem is reduced to a finite matrix diagonalization. In this sense, the naive vacuum world is easier to treat than the correct vacuum world.

We can extend our result for the double harmonic oscillator for the naive vacuum to higher dimensions. The naive vacuum world would then involve polynomials in combinations that are not present in the  $\delta$ -functionals and their derivatives.

#### §6. Naive vacuum world

In this section, we list properties of the naive vacuum world. It is obvious that this world has the following five inappropriate properties from the phenomenological point of view:

1) There is no lowest energy state.

n

2) The "Hilbert space" is not a true Hilbert space, because it is not positive definite. The states with an odd number of negative energy bosons acquire an extra minus sign in the norm-square.

We can introduce boundary conditions to make the model complete. These may be different for negative energy states. As shown in the subsequent paper "Dirac Sea for Bosons. II",<sup>1)</sup> in order to realize an elegant CPT-like symmetry, we propose to employ boundary conditions for the negative energy states such that bound state wave functions diverge.

- 3) We cannot incorporate particles that are their own antiparticles. Therefore, we should consider all particles to possess some charges.
- 4) The naive vacuum world can be viewed as a quantum mechanical system rather than a second quantized field theory. This is because we can think of a finite number of particles, and the second quantized naive vacuum world consists of

spin statistics	$S = \frac{1}{2}, \frac{3}{2}, \cdots$	$S = 0, 1, \cdots$
Fermi-Dirac	$\ \cdots\ ^2 \ge 0$	Indefinite
Bose-Einstein	$\ \cdots\ ^2 \ge 0$	Indefinite

Table I. Spin-statistics theorem for the naive vacuum.

Table II. Spin-statistics theorem for the true vacuum.

statistics	$S = \frac{1}{2}, \frac{3}{2}, \cdots$	$S = 0, 1, \cdots$
Fermi-Dirac	$\ \cdots\ ^2 \ge 0$	Indefinite
Bose-Einstein	Indefinite	$\ \cdots\ ^2 \ge 0$

a superposition of states with various finite numbers of particles.

5) If we accept the negative norm-square, there is no reason to quantize integer spin particles as bosons and half integer spin particles as fermions. Indeed, we find the various possibilities listed in Table I. In this table, we recognize that the well-known spin-statistics theorem is valid only if we require the Hilbert space to be positive definite. It should be noted that in the naive vacuum world with integer spin states, negative norm-squares exist anyway, and thus the spin-statistics theorem does not hold. When we employ the correct vacuum, it becomes possible to avoid a negative norm-square. Then, the problem of an indefinite Hilbert space can be avoided by choosing Bose or Fermi statistics, according to the spin-statistics theorem. This is depicted in Table II.

## §7. Conclusions

We have presented an attempt to extend the idea of the Dirac sea for fermions to the case of bosons. We first considered one second quantization, called the naive vacuum world, in which there exist a few positive and negative energy fermions and bosons but yet no Dirac sea for fermions or bosons. This first picture of the naive vacuum world model has the serious problems with regard to physical properties because it has a bottomless energy spectrum. For bosons, this naive vacuum has even more serious problems because, in addition to negative energies without a lower bound, a state with an odd number of negative energy bosons has a negative normsquare. Hence, there is no actual Hilbert space, but only an indefinite one. At this first step in the boson formulation the inner product for the Fock space is not positive definite. Thus, this first step is completely ruled out from the phenomenological point of view for bosons as well as for fermions. For bosons, this is true for two major reasons: unbounded negative energy and negative norm-square.

However, from the point of view of a theoretical study, this naive vacuum world is very attractive, because the treatment for a few particles is quantum mechanical rather than quantum field theoretical. Furthermore, by locality, the system of several particles becomes free in the neighborhood of almost all configurations, except in the case that some particles meet and interact. We propose the use of this theoretically attractive first step as a theoretical "playground" to gain physical understanding of the real world, which is the second step.

In the present article, we studied the naive vacuum world as a first step. We would like to stress the following major results:

- 1) In the naive vacuum, single particles can exist in position eigenstates, in contrast to the case of particles in "true" relativistic theories.
- 2) The Fock space for bosons is indefinite.
- 3) The Hamiltonian has no lowest energy state. We made some detailed calculations concerning this point.
- 4) We determined the main feature of the wave functionals for bosons, namely, they are derivatives of  $\delta$ -functionals of the complex field multiplied by polynomials in the complex conjugate of the field. These singular wave functionals form a closed class when acted on by operators consisting of polynomials in the creation and annihilation operators. In particular, we worked through the case of one pair consisting of single particle states with opposite momenta.
- 5) In the subsequent paper "Dirac Sea for Bosons. II", we present a CPT-like symmetry. A reduced form of strong reflection provides an extra transformation that is an analytic continuation of the wave function onto another sheet among the  $2^{\frac{1}{2}N(N+1)}$  sheets for the wave function of the N particle system. This sheet structure exists because  $r_{ik}$  is a square root, and therefore it has 2 sheets. For each of the  $\frac{1}{2}N(N+1)$  pairs of particles, there is a dichotomic choice of the sheet, and hence there are  $2^{\frac{1}{2}N(N+1)}$  sheets.

The main point of our present work is to formulate the transition from the naive vacuum of the first step to the correct vacuum of the next step. For fermions, it is known that this can be accomplished by filling the negative energy states. This is what is termed filling the Dirac sea. The corresponding procedure for the case of bosons is that in which from each negative energy single particle state, one boson is removed (that is, minus one boson is added). This removal cannot be carried out in a manner that is as physically realistic as the adding of a fermion, because there is a barrier to be crossed.

We studied this problem using a harmonic oscillator corresponding to a single particle boson state. We replaced the usual Hilbert norm requirement of finiteness by the requirement of the analyticity of the wave function in the entire complex x-plane, except at  $x = \pm \infty$ . The spectrum of this extended harmonic oscillator or the harmonic oscillator with an analytic wave function has an additional series of levels with negative energies, in addition to the usual one. The wave functions with negative energies are of the form of Hermite polynomials multiplied by  $e^{\frac{1}{2}(\beta x)^2}$ .

We note that there is a barrier between the usual states and those with negative excitation numbers, because annihilation and creation operators cannot cause the crossing of the gap between these two sectors. The removal of one particle from an empty negative energy state represents crossing the barrier. Although this cannot be accomplished with a finite number of interactions expressed as a polynomial in creation and annihilation operators, we can still consider such a process. Precisely because of the barrier it is allowed to imagine the possibilities that negative particle numbers could exist without contradicting with experiment. Once the barrier has been crossed to the negative single particle states from the positive ones in a formal way, the model is locked in and those particles cannot return to the positive states. Therefore, it is not a serious problem that the correct vacuum for bosons realizes a *higher* energy than the states with a non-negative number of particles in the negative energy states.

Finally, we mention our underlying motivation for studying in detail the already established second quantization procedure from a different point of view. As mentioned in the introduction, when we consider the quantization of string theories, we may face problems similar to those considered here even in the first quantization, unless we use the light-cone gauge. This was pointed out long ago by Jackiw et al. <sup>8)</sup> Furthermore, there does not seem to exist satisfactory string field theories except for Kaku-Kikkawa's light-cone string field theories. We believe that our bosonic quantization procedure may clarify these problems in the context of string theories.

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