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Charge Quantization Conditions Based on the Atiyah-Singer Index Theorem

Shinichi Deguchi[1,*)] and Kaoru Kitsukawa[2,**)]

[1] *Institute of Quantum Science, College of Science and Technology, Nihon University, Tokyo 101-8308, Japan*
[2] *Graduate School of Quantum Science and Technology, Nihon University, Tokyo 101-8308, Japan*

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Dirac's quantization condition, $eg = n/2$ ($n \in \mathbb{Z}$), and Schwinger's quantization condition, $eg = n$ ($n \in \mathbb{Z}$), for an electric charge e and a magnetic charge g are derived by utilizing the Atiyah-Singer index theorem in two dimensions. The massless Dirac equation on a sphere with a magnetic-monopole background is solved in order to count the number of zero-modes of the Dirac operator.

§1. Introduction

The Dirac quantization condition for an electric charge e and a magnetic charge g reads, in units such that $\hbar = c = 1$,

$$eg = \frac{n}{2}\,, \quad n \in \mathbb{Z}\,. \tag{1·1}$$

Dirac discovered this condition by noting that the phase of the wave function of an electrically charged particle can change by only an integral multiple of 2π under a single rotation of the wave function around a so-called Dirac string.[1),5),6)] Wu and Yang derived the condition (1·1) without reference to the concept of a Dirac string.[2),5),6)] They considered two slightly overlapping hemispheres (often referred to as 'patches' or 'charts') that together surround a magnetic monopole, and introduced a wave function (or section) for an electrically charged particle into each hemisphere. Then the condition (1·1) is obtained by noting that the phases of the two wave functions, which are connected via a gauge transformation, can differ by only an integral multiple of 2π at a given point in the overlapping region. Without explicit use of gauge potentials, Jackiw derived Eq. (1·1) by examining the associative law for translation operators in the case that a magnetic monopole exists.[3)] In this argument, Eq. (1·1) is obtained from the fact that the extra phase (or three-cocycle) arising in a nonassociative algebra must be equal to an integral multiple of 2π to maintain the associativity of translation operators.

In this paper we study an alternative approach to deriving Eq. (1·1) using the Atiyah-Singer index theorem in two dimensions.[7)–11),15),16)] As we see below, in order to obtain Eq. (1·1) in this approach, it is only necessary to solve a simple

*) E-mail: deguchi@phys.cst.nihon-u.ac.jp
**) E-mail: kaworu@phys.cst.nihon-u.ac.jp

Dirac equation in two dimensions and to formally count the number of zero-modes of the Dirac operator. In addition to the Dirac quantization condition (1·1), our approach yields the Schwinger quantization condition,[4)–6)]

$$eg = n\,, \quad n \in \mathbb{Z}\,. \tag{1·2}$$

This condition was discovered by Schwinger in his study of a relativistic quantum field theory of electric and magnetic charges.[4)] There, it is verified that the relativistic invariance of the theory is maintained only when the condition (1·2) is satisfied. The present paper treats the conditions (1·1) and (1·2) in a unified manner; each is obtained by fixing a parameter contained in the gauge potential at a suitable value.

This paper is organized as follows. Section 2 presents the Atiyah-Singer index theorem in two dimensions. In §3, a massless Dirac equation with a magnetic-monopole background is solved to find its regular solutions, namely the zero-modes of the Dirac operator. In §4, the number of zero-modes is counted to derive the charge quantization conditions (1·1) and (1·2). Section 5 contains concluding remarks. The Appendix is devoted to a simple proof of the Atiyah-Singer index theorem in two dimensions so that this paper may be self-contained.

§2. Atiyah-Singer index theorem in two dimensions

Let $\mathcal{M}$ be a compact, oriented, two-dimensional Riemannian manifold without boundary. Assuming that $\mathcal{M}$ possesses a spin structure, we consider a self-adjoint Dirac operator $i\not{D}$ containing a hermitian Yang-Mills connection A that takes values in the Lie algebra of a compact, simple Lie group $\mathcal{G}$. The self-adjointness of $i\not{D}$ is defined with respect to the natural L^2 inner product on spinors, which is given by integration over $\mathcal{M}$ involving the Riemannian surface-element on $\mathcal{M}$. In terms of local coordinates (q^α) $(\alpha = 1, 2)$ on $\mathcal{M}$, the Dirac operator can be expressed as

$$i\not{D} \equiv i\sigma_a e_a{}^\alpha D_\alpha\,, \tag{2·1}$$

with

$$D_\alpha \equiv \sigma_0 \partial_\alpha + \frac{i}{2}\omega_\alpha \sigma_3 - ieA_\alpha \sigma_0\,. \tag{2·2}$$

Here $\partial_\alpha \equiv \partial/\partial q^\alpha$, $e_a{}^\alpha$ $(a = 1, 2)$ is an inverse zweibein related to the Riemannian metric $g_{\alpha\beta}$ on $\mathcal{M}$, ω_α is a spin connection in two dimensions, e is an electric charge, and σ_0 denotes the 2×2 unit matrix, while σ_a and σ_3 denote the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2·3}$$

Now consider the positive chirality zero-modes $\{\varphi^+_{0,\nu_+}\}$ $(\nu_+ = 1, \ldots, \mathfrak{n}_+)$ and the negative chirality zero-modes $\{\varphi^-_{0,\nu_-}\}$ $(\nu_- = 1, \ldots, \mathfrak{n}_-)$ of $i\not{D}$, characterized together by

$$i\not{D}\varphi^\pm_{0,\nu_\pm}(q) = 0\,, \tag{2·4}$$

$$\sigma_3 \varphi^\pm_{0,\nu_\pm} = \pm\varphi^\pm_{0,\nu_\pm}\,. \tag{2·5}$$

Here, $\mathfrak{n}_+$ ($\mathfrak{n}_-$) denotes the number of positive (negative) chirality zero-modes. Then the Atiyah-Singer index theorem in two dimensions reads[7)–11),15),16)]

$$\mathfrak{n}_+ - \mathfrak{n}_- = \frac{e}{4\pi}\int_{\mathcal{M}} d^2q\,\mathrm{tr}\,\varepsilon^{\alpha\beta}F_{\alpha\beta}\,, \tag{2·6}$$

where $\varepsilon^{\alpha\beta}$ ($\varepsilon^{12}=1$) is the contravariant Levi-Civita tensor *density* in two dimensions, and $F_{\alpha\beta}$ is the field strength of A_α,

$$F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha - ie[A_\alpha, A_\beta]\,. \tag{2·7}$$

The notation "tr" indicates the trace over the generators of $\mathcal{G}$. A simple proof of the Atiyah-Singer index theorem in two dimensions is given in the Appendix.

§3. Massless Dirac equation with a monopole background

Let us consider the case in which $\mathcal{M}=S^2$ and $\mathcal{G}=U(1)$. Then the Atiyah-Singer index theorem (2·6) reads

$$\mathfrak{n}_+ - \mathfrak{n}_- = \frac{e}{4\pi}\int_{S^2} d^2q\,\varepsilon^{\alpha\beta}F_{\alpha\beta}\,, \tag{3·1}$$

with $F_{\alpha\beta}=\partial_\alpha A_\beta - \partial_\beta A_\alpha$. In terms of spherical coordinates, $(q^1,q^2)=(\theta,\phi)$ ($0\le\theta\le\pi$, $0\le\phi<2\pi$), on S^2 of radius r, the diagonal zweibein ${e_\alpha}^a$ and its inverse ${e_a}^\alpha$ are given by

$$(e_{\alpha a}) = \mathrm{diag}(r,\, r\sin\theta)\,, \quad ({e_a}^\alpha) = \mathrm{diag}(r^{-1},\, r^{-1}\sin^{-1}\theta)\,, \tag{3·2}$$

and thus the Riemannian metric takes the standard form $(g_{\alpha\beta}) = (e_{\alpha a}e_{\beta a}) = \mathrm{diag}(r^2,\, r^2\sin^2\theta)$. The associated spin connection ω_α is found by using the torsion-free condition[*)] and Eq. (3·2) to be

$$\begin{aligned}\omega_\alpha &= \frac{1}{2}e_{\alpha a}\epsilon_{bc}\big({e_a}^\beta {e_b}^\gamma \partial_\beta e_{\gamma c} + {e_c}^\beta {e_a}^\gamma \partial_\beta e_{\gamma b} - {e_b}^\beta {e_c}^\gamma \partial_\beta e_{\gamma a}\big)\\ &= -\delta_{\alpha 2}\cos\theta\,,\end{aligned} \tag{3·3}$$

where ϵ_{bc} ($\epsilon_{12}=1$) is the Levi-Civita tensor in two-dimensional flat space.

In addition to these quantities, we now choose the gauge potential A_α to be that describing a monopole configuration,[5),6)]

$$A_\alpha = \delta_{\alpha 2}g(\kappa-\cos\theta) = e_{\alpha 2}\frac{g(\kappa-\cos\theta)}{r\sin\theta}\,, \tag{3·4}$$

where $\kappa=-1,0,1$, and g denotes a magnetic charge. The field strength $F_{\alpha\beta}$ of this potential is actually the static magnetic field due to a point magnetic monopole of strength g situated at the point $r=0$:

$$F_{\alpha\beta} = \varepsilon_{\alpha\beta}g\sin\theta = \epsilon_{ab}e_{\alpha a}e_{\beta b}\frac{g}{r^2}\,, \tag{3·5}$$

[*)] The torsion-free condition is given by ${\Gamma_{\beta\gamma}}^\alpha = {\Gamma_{\gamma\beta}}^\alpha$, with ${\Gamma_{\beta\gamma}}^\alpha$ being the affine connection. The first line of Eq. (3·3) is obtained by applying the torsion-free condition to Eq. (A·10) in the Appendix.

where $\varepsilon_{\alpha\beta}$ ($\varepsilon_{12}=1$) is the covariant Levi-Civita tensor *density* in two dimensions. From the second expression in Eq. (3·4), it is seen that when $\kappa=1$, $A_a \equiv e_a{}^\alpha A_\alpha$ has a singularity only at the south pole, (π,ϕ), on S^2, while when $\kappa=-1$, A_a has a singularity only at the north pole, $(0,\phi)$, on S^2. This situation holds for all spheres of all possible radii r ($0 \le r < \infty$), and hence it follows that, as r runs from 0 to ∞, the singularity at each pole comes to form a set of singularities that constitutes a semi-infinite string whose endpoint is at $r=0$. Such a semi-infinite string is known as a Dirac string, and the cases $\kappa=1$ and $\kappa=-1$ are sometimes referred to as the *Dirac formalism*.[5] By contrast, when $\kappa=0$, A_a has singularities at both the north and south poles, in accordance with space-reflection considerations. As a result, as r runs from 0 to ∞, these two singularities come to form a set of singularities that constitutes an infinite string with no ends. The case $\kappa=0$ is sometimes referred to as the *Schwinger formalism*,[5] because Schwinger studied the case $\kappa=0$ in particular to maintain compatibility of the magnetic-charge concept with the principles of relativistic quantum field theory.[4] Essentially, the Dirac formalism treats a semi-infinite string and yields the charge quantization condition $eg=n/2$, whereas the Schwinger formalism treats an infinite string and yields the charge quantization condition $eg=n$. In the present paper, we consider both the Dirac formalism and the Schwinger formalism to derive the corresponding charge-quantization conditions by utilizing the Atiyah-Singer index theorem in two dimensions.

Substituting Eqs. (3·2)–(3·4) into Eq. (2·1), we can write the massless Dirac equation $i\not{D}\varphi=0$ in the following form:[12]

$$\frac{i}{r}\begin{pmatrix} 0 & \nabla_\theta - \dfrac{i}{\sin\theta}\nabla_\phi \\ \nabla_\theta + \dfrac{i}{\sin\theta}\nabla_\phi & 0 \end{pmatrix}\begin{pmatrix} u^+(\theta,\phi) \\ u^-(\theta,\phi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3·6}$$

with

$$\nabla_\theta \equiv \frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta\,, \quad \nabla_\phi \equiv \frac{\partial}{\partial\phi} - ieg(\kappa-\cos\theta)\,. \tag{3·7}$$

Equation (3·6) can be separated into two differential equations by substituting $u^\pm(\theta,\phi)=v^\pm(\theta)w^\pm(\phi)$ into it. The resulting differential equation in ϕ can be immediately solved, and we obtain the normalized solutions $w^\pm_{m_\pm}(\phi)=(2\pi)^{-1/2}e^{im_\pm\phi}$. Here m_+ and m_- are half-integers, that is, $m_+, m_- = \pm1/2, \pm3/2, \ldots$, because the spinor field φ has to change sign under a 2π rotation in ϕ. The differential equation in θ is thus obtained as

$$\left[\frac{d}{d\theta} + \left(\frac{1}{2}\mp eg\right)\cot\theta \mp \frac{m_\pm - eg\kappa}{\sin\theta}\right]v^\pm(\theta)=0\,. \tag{3·8}$$

The normalized solutions of this equation are readily found to be

$$v^\pm_{m_\pm}(\theta) = c^\pm_{m_\pm}\left(\sin\frac{\theta}{2}\right)^{p^\pm_{m_\pm}}\left(\cos\frac{\theta}{2}\right)^{q^\pm_{m_\pm}}, \tag{3·9}$$

where

$$p^{\pm}_{m_\pm} \equiv \pm\{m_\pm - eg(\kappa-1)\} - \frac{1}{2}\,, \tag{3·10}$$

$$q^{\pm}_{m_\pm} \equiv \mp\{m_\pm - eg(\kappa+1)\} - \frac{1}{2}\,, \tag{3·11}$$

and $c^{\pm}_{m_\pm}$ is the normalization constant applicable to $p^{\pm}_{m_\pm}, q^{\pm}_{m_\pm} > -1$:

$$c^{\pm}_{m_\pm} = \sqrt{\frac{\Gamma(p^{\pm}_{m_\pm} + q^{\pm}_{m_\pm} + 2)}{2\Gamma(p^{\pm}_{m_\pm}+1)\Gamma(q^{\pm}_{m_\pm}+1)}}\,. \tag{3·12}$$

The solution $v^{\pm}_{m_\pm}$ diverges at neither $\theta = 0$ nor π if and only if $p^{\pm}_{m_\pm}, q^{\pm}_{m_\pm} \geq 0$. In this case, $v^{\pm}_{m_\pm}$ has a definite normalization constant $c^{\pm}_{m_\pm}$. The conditions $p^{+}_{m_+}, q^{+}_{m_+} \geq 0$ necessary for $v^{+}_{m_+}$ to be finite at $\theta = 0, \pi$ can be expressed as

$$\frac{1}{2} + eg(\kappa-1) \leq m_+ \leq -\frac{1}{2} + eg(\kappa+1)\,, \tag{3·13}$$

implying $eg \geq 1/2$, while the conditions $p^{-}_{m_-}, q^{-}_{m_-} \geq 0$ necessary for $v^{-}_{m_-}$ to be finite at $\theta = 0, \pi$ can be expressed as

$$\frac{1}{2} + eg(\kappa+1) \leq m_- \leq -\frac{1}{2} + eg(\kappa-1)\,, \tag{3·14}$$

implying $eg \leq -1/2$. Therefore the conditions $p^{+}_{m_+}, q^{+}_{m_+} \geq 0$ and the conditions $p^{-}_{m_-}, q^{-}_{m_-} \geq 0$ are never satisfied simultaneously with a given eg. For this reason, the possible *regular* solutions of $i\not{D}\varphi = 0$ are restricted to

$$\varphi^{+}_{m_+} = \begin{pmatrix} u^{+}_{m_+} \\ 0 \end{pmatrix} \text{ for } eg \geq 1/2\,, \quad \varphi^{-}_{m_-} = \begin{pmatrix} 0 \\ u^{-}_{m_-} \end{pmatrix} \text{ for } eg \leq -1/2\,, \tag{3·15}$$

with $u^{\pm}_{m_\pm} \equiv v^{\pm}_{m_\pm} w^{\pm}_{m_\pm}$. They satisfy the orthonormality relation

$$\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \left(\varphi^{\pm}_{m_\pm}\right)^\dagger \varphi^{S}_{m'_S} = \begin{cases} \delta_{m_\pm m'_\pm}\,, & (S = \pm) \\ 0\,. & (S = \mp) \end{cases} \tag{3·16}$$

Each of these solutions, characterized by $m_\pm$, has a definite chirality, satisfying Eq. (2·5) as well as Eq. (2·4). Since $\sigma_3\varphi^{\pm}_{m_\pm} = \pm\varphi^{\pm}_{m_\pm}$, the solutions $\{\varphi^{+}_{m_+}\}$ are recognized as the positive chirality zero-modes, while the solutions $\{\varphi^{-}_{m_-}\}$ are recognized as the negative chirality zero-modes. It is now obvious that $\mathfrak{n}_- = 0$ for $eg \geq 1/2$, $\mathfrak{n}_+ = 0$ for $eg \leq -1/2$, and $\mathfrak{n}_+ = \mathfrak{n}_- = 0$ for $|eg| < 1/2$. When $e = 0$, the Dirac operator $i\not{D}$, of course, has no zero-modes. This illustrates the Lichnerowicz vanishing theorem.[13] (Also see under Eq. (A·19) in the Appendix.)

§4. Count of zero-modes and derivation of charge quantization conditions

We now count the number of zero-modes of $i\not{D}$ to find $\mathfrak{n}_+$ for $eg \geq 1/2$ and $\mathfrak{n}_-$ for $eg \leq -1/2$, and then derive the charge quantization conditions.

First, consider the case $\kappa = 1$, one case of the Dirac formalism. When eg ($\geq 1/2$) is in the interval $n/2 \leq eg < (n+1)/2$ with $n = 1, 2, 3, \ldots$, the allowed values of m_+ can be seen immediately from Eq. (3·13) to be $m_+ = 1/2, 3/2, \ldots, (2n-1)/2$. (Recall here that m_+ and m_- take half-integer values.) This implies that the number of regular solutions of $i\not{D}\varphi = 0$ is n, and it follows that $\mathfrak{n}_+ = n$. When eg ($\leq -1/2$) is in the interval $-(n+1)/2 < eg \leq -n/2$ with $n = 1, 2, 3, \ldots$, the allowed values of m_- are found from Eq. (3·14) to be $m_- = -1/2, -3/2, \ldots, -(2n-1)/2$, and it follows that $\mathfrak{n}_- = n$. The total magnetic flux due to a point magnetic monopole is obtained from Eq. (3·5) as

$$\frac{1}{2}\int_{S^2} d\theta d\phi\, \varepsilon^{\alpha\beta} F_{\alpha\beta} = g\int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = 4\pi g\,. \tag{4·1}$$

Thus, in this case, the Atiyah-Singer index theorem (3·1) leads to the following relations: $n = 2eg$ for $eg \geq 1/2$, $-n = 2eg$ for $eg \leq -1/2$, and $0 = 2eg$ for $|eg| < 1/2$. These are compatible with the choice of the intervals for eg mentioned above. More precisely, because the relation $n = 2eg$ ($-n = 2eg$) was derived for eg assumed to be in the interval $n/2 \leq eg < (n+1)/2$ ($-(n+1)/2 < eg \leq -n/2$) closed at $eg = n/2$ ($eg = -n/2$), the compatibility of the relation and the interval is maintained. The three relations found here are brought together in the form $eg = n/2$ with $n \in \mathbb{Z}$. This is precisely the Dirac quantization condition.

Next, consider the case $\kappa = -1$, which is another case of the Dirac formalism, though it is a mirror image of the case $\kappa = 1$. When eg ($\geq 1/2$) is in the interval $n/2 \leq eg < (n+1)/2$ with $n = 1, 2, 3, \ldots$, the allowed values of m_+ are found from Eq. (3·13) to be $m_+ = -1/2, -3/2, \ldots, -(2n-1)/2$, so that $\mathfrak{n}_+ = n$. When eg ($\leq -1/2$) is in the interval $-(n+1)/2 < eg \leq -n/2$ with $n = 1, 2, 3, \ldots$, the allowed values of m_- are found from Eq. (3·14) to be $m_- = 1/2, 3/2, \ldots, (2n-1)/2$, so that $\mathfrak{n}_- = n$. Therefore the possible values of $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are the same as those in the case $\kappa = 1$, as might be expected; the Dirac quantization condition follows again, maintaining the compatibility with the choice of the intervals for eg.

Finally, consider the case $\kappa = 0$, i.e., that of the Schwinger formalism. When eg is in the interval $1/2 \leq eg < 1$, there are no allowed values of m_+, as can be seen from Eq. (3·13), so that $\mathfrak{n}_+ = 0$. When eg (≥ 1) is in the interval $n \leq eg < n+1$ with $n = 1, 2, 3, \ldots$, the allowed values of m_+ can be seen immediately from Eq. (3·13) to be $m_+ = \pm 1/2, \pm 3/2, \ldots, \pm(2n-1)/2$, and it follows that $\mathfrak{n}_+ = 2n$. When eg is in the interval $-1 < eg \leq -1/2$, there are no allowed values of m_-, as can be seen from Eq. (3·14), so that $\mathfrak{n}_- = 0$. When eg (≤ -1) is in the interval $-(n+1) < eg \leq -n$ with $n = 1, 2, 3, \ldots$, the allowed values of m_- are found from Eq. (3·14) to be $m_- = \pm 1/2, \pm 3/2, \ldots, \pm(2n-1)/2$, and it follows that $\mathfrak{n}_- = 2n$. Thus, in the case $\kappa = 0$, the Atiyah-Singer index theorem (3·1) leads to the following relations: $2n = 2eg$ for $eg \geq 1$, $-2n = 2eg$ for $eg \leq -1$, and $0 = 2eg$ for $|eg| < 1$. These are also compatible with the choice of the intervals for eg. The three relations are brought together in the form $eg = n$ with $n \in \mathbb{Z}$. This is precisely the Schwinger quantization condition.

§5. Conclusions

We have proposed a novel method for deriving the charge quantization conditions that are caused by the presence of a point magnetic monopole situated at the origin in three-dimensional Euclidean space. Although our investigation considers four-dimensional spacetime, the problem itself is reduced to one in two-dimensional space by virtue of the static and scale-independent conditions of the system [see Eqs. (3·4) and (3·5)]. For this reason, it was possible to examine the charge quantization conditions by placing two-component static chiral spinor fields on a sphere at whose centre there exists a point magnetic monopole.

Employing such a system, we derived both the Dirac quantization condition, $eg = n/2$ ($n \in \mathbb{Z}$), and the Schwinger quantization condition, $eg = n$ ($n \in \mathbb{Z}$), by utilizing the Atiyah-Singer index theorem in two dimensions. These quantization conditions can be understood as the conditions that the Atiyah-Singer index theorem in two dimensions holds without giving rise to the index defect. In other words, the charge quantization conditions may be regarded as the consistency conditions to be satisfied when zero-modes of the Dirac operator are allowed to exist on a sphere at whose centre there lies a point magnetic monopole.

As we have seen, the Dirac quantization condition, $eg = n/2$, is found when κ in Eq. (3·4) is equal to 1 or -1, while the Schwinger quantization condition, $eg = n$, is found when $\kappa = 0$. These two conditions were obtained by solving the simple Dirac equation (3·6) and by formally counting the number of zero-modes of the Dirac operator. In our procedure, the difference between the Dirac and Schwinger quantization conditions simply results from the fact that the number of zero-modes in the Schwinger formalism is twice that in the Dirac formalism for a fixed integer n.

Our approach requires neither a careful treatment of a string singularity in the gauge potential nor the concept of patches (or charts) and sections. In fact, in this paper, we have treated neither the gauge transformation of wave functions (or sections) nor of gauge potentials.

Finally, note that if we take the charge quantization conditions $eg = n/2$ and $eg = n$ as given by other approaches, the argument presented in this paper may be understood as an illustration or verification of the Atiyah-Singer index theorem in a particular case.

Appendix

In this appendix, we prove the Atiyah-Singer index theorem in two dimensions, expressed by Eq. (2·6), by using the heat kernel method.[14),15),17),19),20)]

Let us consider the eigenvalue equation for the Dirac operator $i\not{D}$ given in Eq. (2·1),

$$i\not{D}\varphi_N(q) = \lambda_n \varphi_N(q)\,, \tag{A·1}$$

with an eigenvalue λ_n and an eigenfunction φ_N. Here, N is a collective index representing the pair of indices (n, ν), where n labels the eigenvalues of $i\not{D}$ in such a way

that $n \neq n' \rightleftarrows \lambda_n \neq \lambda_{n'}$, while ν distinguishes between the degenerate eigenfunctions of $i\not{D}$ corresponding to the same eigenvalue λ_n. Because $i\not{D}$ is self-adjoint, as can be proven by using the premise that $\mathcal{M}$ has no boundary, the eigenvalue λ_n is real and the eigenfunctions $\{\varphi_N\}$ form an orthonormal set:

$$\int_{\mathcal{M}} d^2q\sqrt{\mathfrak{g}(q)}\,\varphi_N^\dagger(q)\varphi_{N'}(q) = \delta_{NN'}\,, \tag{A·2}$$

with $\sqrt{\mathfrak{g}} \equiv |\det(e_a{}^\alpha)|^{-1}$ and $\delta_{NN'} \equiv \delta_{nn'}\delta_{\nu\nu'}$. (Because $\mathcal{M}$ is compact, the normalization of φ_N is always possible.) The following completeness condition can be imposed on $\{\varphi_N\}$:

$$\sum_N \varphi_N(q)\varphi_N^\dagger(q') = \delta^2(q,q')\sigma_0\,. \tag{A·3}$$

Here, $\delta^2(q,q')$ is a generalized delta-function on two-dimensional curved space,[14),16)] satisfying

$$f(q) = \int_{\mathcal{M}} d^2q'\sqrt{\mathfrak{g}(q')}\,\delta^2(q,q')f(q') \tag{A·4}$$

for an arbitrary smooth function $f(q)$ on $\mathcal{M}$.[*)]

Let us now consider the function

$$\mathcal{A}(q) \equiv \sum_N \varphi_N^\dagger(q)\sigma_3\varphi_N(q)\,. \tag{A·5}$$

This function is not well-defined, because it involves an infinite sum of the functions $\varphi_N^\dagger\sigma_3\varphi_N$ at the same point (q^α). Hence we have to carry out a suitable regularization of $\mathcal{A}(q)$ in order to evaluate it. For this purpose, we insert a Gaussian cutoff with a positive parameter τ into the right-hand side of Eq. (A·5) and rewrite the regularized form by using the eigenvalue equation (A·1):[15),16)]

$$\begin{aligned}
\mathcal{A}_{\rm reg}(q) &\equiv \lim_{\tau\searrow 0}\sum_N \varphi_N^\dagger(q)\sigma_3 e^{-\tau\lambda_N^2}\varphi_N(q)\\
&= \lim_{\tau\searrow 0}\sum_N \varphi_N^\dagger(q)\sigma_3\exp\bigl[-\tau(i\not{D})^2\bigr]\varphi_N(q)\\
&= \lim_{\tau\searrow 0}\mathrm{Tr}\Bigl(\sigma_3\sum_N\bigl\{\exp\bigl[-\tau(i\not{D})^2\bigr]\varphi_N(q)\bigr\}\varphi_N^\dagger(q)\Bigr)\\
&= \lim_{\tau\searrow 0}\lim_{q'\to q}\mathrm{Tr}\bigl(\sigma_3 G(q,q',\tau)\bigr)\,,
\end{aligned} \tag{A·6}$$

where the notation "Tr" indicates the trace taken over the Pauli matrices and over the generators of the Lie group $\mathcal{G}$. The two-point function G is defined by

$$G(q,q',\tau) \equiv \sum_N\bigl\{\exp\bigl[-\tau(i\not{D})^2\bigr]\varphi_N(q)\bigr\}\varphi_N^\dagger(q')\,. \tag{A·7}$$

*) On the right-hand side of Eq. (A·3), the appearance of the unit matrix T_0 in a representation space of the Lie group $\mathcal{G}$, satisfying $(T_0\otimes\sigma_0)\varphi_N(q) = \varphi_N(q)$, is understood. The appearance of this matrix is also understood in some equations in this paper.

It is readily shown that G satisfies the so-called heat equation,

$$-\frac{\partial}{\partial\tau}G(q,q',\tau) = (i\not{D})^2 G(q,q',\tau)\,, \tag{A·8}$$

as well as the initial condition due to the completeness condition (A·3),

$$G(q,q',0) = \delta^2(q,q')\sigma_0\,. \tag{A·9}$$

The function $G(q,q',\tau)$ is often referred to as the heat kernel.

Before trying to solve Eq. (A·8) supplemented with Eq. (A·9), we calculate the square of the Dirac operator $i\not{D}$. To this end, it is convenient to introduce the affine connection $\Gamma_{\alpha\beta}{}^{\gamma}$, which appears, for instance, in the condition describing the parallel transport on $\mathcal{M}$,

$$\partial_\beta e_a{}^\alpha + \Gamma_{\beta\gamma}{}^\alpha e_a{}^\gamma + \epsilon_{ab}\omega_\beta e_b{}^\alpha = 0\,, \tag{A·10}$$

where ϵ_{ab} ($\epsilon_{12}=1$) is the Levi-Civita tensor in two-dimensional flat space. Because $\mathcal{M}$ is a Riemannian manifold and thus is torsion-free, the affine connection is symmetric in the lower indices: $\Gamma_{\beta\gamma}{}^\alpha = \Gamma_{\gamma\beta}{}^\alpha$. From this relation and Eq. (A·10), the affine connection is determined in terms of the inverse metric $g^{\alpha\beta} = \delta_{ab}e_a{}^\alpha e_b{}^\beta$ and the derivative of the metric $g_{\alpha\beta} = \delta_{ab}e_{\alpha a}e_{\beta b}$ to be

$$\Gamma_{\alpha\beta}{}^\gamma = \frac{1}{2}g^{\gamma\delta}(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta})\,. \tag{A·11}$$

Also, the quadratic operator $(i\not{D})^2$ can be calculated, using the fundamental properties of the Pauli matrices $[\sigma_i,\sigma_j] = 2i\epsilon_{ijk}\sigma_k$ and $\{\sigma_i,\sigma_j\} = 2\delta_{ij}$, as

$$(i\not{D})^2 = -\sigma_0 g^{\alpha\beta}(D_\alpha D_\beta - \Gamma_{\alpha\beta}{}^\gamma D_\gamma) - \frac{1}{4}[\sigma^\alpha,\sigma^\beta][D_\alpha,D_\beta]\,, \tag{A·12}$$

where $\sigma^\alpha \equiv \sigma_a e_a{}^\alpha$. The commutator of the covariant derivatives is easily calculated from Eq. (2·2) as

$$[D_\alpha, D_\beta] = \frac{i}{2}\Omega_{\alpha\beta}\sigma_3 - ieF_{\alpha\beta}\sigma_0\,, \tag{A·13}$$

with

$$\Omega_{\alpha\beta} \equiv \partial_\alpha\omega_\beta - \partial_\beta\omega_\alpha\,, \tag{A·14}$$

$$F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha - ie[A_\alpha, A_\beta]\,. \tag{A·15}$$

Substituting Eq. (A·13) into Eq. (A·12), we can proceed with calculating the quadratic operator, and we finally obtain

$$(i\not{D})^2 = -\sigma_0\Delta + \frac{1}{4}R\sigma_0 - \frac{1}{2}e\epsilon^{\alpha\beta}F_{\alpha\beta}\sigma_3\,, \tag{A·16}$$

with

$$\Delta \equiv g^{\alpha\beta}(D_\alpha D_\beta - \Gamma_{\alpha\beta}{}^\gamma D_\gamma) = \frac{1}{\sqrt{\mathfrak{g}}}D_\alpha\sqrt{\mathfrak{g}}\,g^{\alpha\beta}D_\beta\,, \tag{A·17}$$

$$R \equiv \epsilon^{\alpha\beta}\Omega_{\alpha\beta}\,, \tag{A·18}$$

$$\epsilon^{\alpha\beta} \equiv \epsilon_{ab}e_a{}^\alpha e_b{}^\beta = \frac{1}{\sqrt{\mathfrak{g}}}\varepsilon^{\alpha\beta}\,. \tag{A·19}$$

The operator Δ, which is negative-semidefinite, is the Laplacian taken into account the covariance under general coordinate transformations as well as that under local Lorentz and the extra gauge transformations. The function R is the scalar curvature of $\mathcal{M}$. Equation (A·16) implies that for manifolds with positive curvature, there exist no zero modes of the Dirac operator $i\not{D}$ if $e = 0$. This is known as the Lichnerowicz vanishing theorem.[13)]

Now we return to the heat equation (A·8). Because the double limit $q' \to q$, $\tau \searrow 0$ is taken in Eq. (A·6), only the asymptotic form of $G(q, q', \tau)$ in this limit is necessary for the evaluation of $\mathcal{A}_{\rm reg}(q)$. In order to simply derive this asymptotic form, we employ the Riemann normal coordinates with the origin at (q'^{α}), characterized by $\partial_\gamma g_{\alpha\beta}(q') = 0$ [or equivalently $\Gamma_{\alpha\beta}{}^{\gamma}(q') = 0$] and $g_{\alpha\beta}(q') = \delta_{\alpha\beta}$.[16)–18)] Then, in an infinitesimal neighborhood of (q'^{α}), denoted by $\mathcal{U}_{q'}$, the metric $g_{\alpha\beta}$ at $(q^{\alpha}) \in \mathcal{U}_{q'}$ and its inverse $g^{\alpha\beta}$ can be expressed in the forms of their respective Taylor series:

$$g_{\alpha\beta}(q) = \delta_{\alpha\beta} - \frac{1}{3} R_{\alpha\gamma\beta\delta}(q')(q^{\gamma} - q'^{\gamma})(q^{\delta} - q'^{\delta}) + O\big((q - q')^3\big)\,, \tag{A·20}$$

$$g^{\alpha\beta}(q) = \delta^{\alpha\beta} + \frac{1}{3} R^{\alpha}{}_{\gamma}{}^{\beta}{}_{\delta}(q')(q^{\gamma} - q'^{\gamma})(q^{\delta} - q'^{\delta}) + O\big((q - q')^3\big)\,. \tag{A·21}$$

Here, $R_{\alpha\gamma\beta}{}^{\delta}$ denotes the Riemann tensor,

$$\begin{aligned} R_{\alpha\gamma\beta}{}^{\delta} &= -\partial_\alpha \Gamma_{\gamma\beta}{}^{\delta} + \partial_\gamma \Gamma_{\alpha\beta}{}^{\delta} - \Gamma_{\alpha\epsilon}{}^{\delta}\Gamma_{\gamma\beta}{}^{\epsilon} + \Gamma_{\gamma\epsilon}{}^{\delta}\Gamma_{\alpha\beta}{}^{\epsilon} \\ &= \Omega_{\alpha\gamma}\epsilon_{ab} e_{\beta a} e_b{}^{\delta}\,, \end{aligned} \tag{A·22}$$

with the (anti)symmetric properties $R_{\alpha\gamma\beta\delta} = -R_{\gamma\alpha\beta\delta} = R_{\beta\delta\alpha\gamma}$. All the expansion coefficients in the two Taylor series (A·20) and (A·21) are written in terms of the Riemann tensor and its covariant derivatives of first and higher orders that are all evaluated at (q'^{α}).[18)] The Taylor expansion of $\Gamma_{\alpha\beta\gamma}(q)$ about (q'^{α}) is found from Eqs. (A·11) and (A·20) to be

$$\Gamma_{\alpha\beta\gamma}(q) = \frac{1}{3}\big(R_{\alpha\gamma\beta\delta}(q') + R_{\beta\gamma\alpha\delta}(q')\big)(q^{\delta} - q'^{\delta}) + O\big((q - q')^2\big)\,. \tag{A·23}$$

After substituting Eqs. (A·21) and (A·23) into Eq. (A·17), the covariant Laplacian Δ can be written in the following form:

$$\begin{aligned} \Delta = &\left(\delta^{\alpha\beta} + \frac{1}{3} R^{\alpha}{}_{\gamma}{}^{\beta}{}_{\delta}(q')(q^{\gamma} - q'^{\gamma})(q^{\delta} - q'^{\delta})\right) D_\alpha D_\beta \\ &- \frac{2}{3} R_{\alpha}{}^{\gamma\alpha}{}_{\delta}(q')(q^{\delta} - q'^{\delta}) D_\gamma + O\big((q - q')^3\big)\,. \end{aligned} \tag{A·24}$$

For the special case $(i\not{D})^2 = -\sigma_0 \delta^{\alpha\beta}\partial_\alpha\partial_\beta$, supplemented with the condition

$$G(q, q', 0) = \delta^2(q, q')\sigma_0 = \delta(q^1 - q'^1)\delta(q^2 - q'^2)\sigma_0\,, \quad (q^{\alpha}) \in \mathcal{U}_{q'}\,, \tag{A·25}$$

the heat equation (A·8) has the Gaussian solution

$$\widetilde{G}(q, q', \tau) = \frac{1}{4\pi\tau}\sigma_0 \exp\left[-\frac{1}{4\tau}\delta_{\alpha\beta}(q^{\alpha} - q'^{\alpha})(q^{\beta} - q'^{\beta})\right]. \tag{A·26}$$

We are now interested in the behavior of $G(q,q',\tau)$ for extremely small τ. In that case, the following expansion is valid:

$$G(q,q',\tau)=\widetilde{G}(q,q',\tau)\sum_{k=0}^{\infty}a_k(q,q')\tau^k\,,\quad 0<\tau\ll 1\,, \tag{A·27}$$

where $a_k(q,q')$ $(k=0,1,2,\ldots)$ are the so-called De Witt–Seeley coefficients.[14] Substituting Eq. (A·27) into the heat equation (A·8) with the covariant Laplacian (A·24), we obtain the recursion relations

$$\begin{aligned}&\left\{k+(q^\alpha-q'^\alpha)D_\alpha-\frac{1}{6}R^\gamma{}_{\alpha\gamma\beta}(q')(q^\alpha-q'^\alpha)(q^\beta-q'^\beta)+O\big((q-q')^3\big)\right\}a_k(q,q')\\&=-\left\{(i\not{D})^2+O\big((q-q')^3\big)\right\}a_{k-1}(q,q')\,,\qquad k=1,2,3,\ldots\end{aligned} \tag{A·28}$$

and

$$\left\{(q^\alpha-q'^\alpha)D_\alpha-\frac{1}{6}R^\gamma{}_{\alpha\gamma\beta}(q')(q^\alpha-q'^\alpha)(q^\beta-q'^\beta)+O\big((q-q')^3\big)\right\}a_0(q,q')=0\,. \tag{A·29}$$

In the derivation of these relations, some irrelevant terms vanish by virtue of the antisymmetric property of the indices of the Riemann tensor. Then, taking the covariant derivative of Eq. (A·29) yields

$$\left\{D_\beta+(q^\gamma-q'^\gamma)\left(D_\beta D_\gamma-\frac{1}{3}R^\delta{}_{\beta\delta\gamma}(q')\right)+O\big((q-q')^2\big)\right\}a_0(q,q')=0\,. \tag{A·30}$$

Next, applying D_α to this formula and contracting the indices α and β, we have

$$\left\{g^{\alpha\beta}D_\alpha D_\beta-\frac{1}{6}R(q')+O(q-q')\right\}a_0(q,q')=0\,, \tag{A·31}$$

where $R=g^{\alpha\beta}R^\gamma{}_{\alpha\gamma\beta}$ is the scalar curvature which, of course, is identical to R defined in Eq. (A·18), as can be seen from Eq. (A·22).

It is easy to see that the condition (A·25) yields the normalization condition

$$\lim_{q'\to q}a_0(q,q')=\sigma_0\,. \tag{A·32}$$

In addition, it follows from Eqs. (A·30) and (A·31) that

$$\lim_{q'\to q}D_\beta a_0(q,q')=0\,, \tag{A·33}$$

$$\lim_{q'\to q}g^{\alpha\beta}D_\alpha D_\beta a_0(q,q')=\frac{1}{6}R\sigma_0\,. \tag{A·34}$$

Taking the limit $q'\to q$ in the recursion relation (A·28) with $k=1$ and recalling Eqs. (A·16) and (A·17), we have

$$\begin{aligned}&\lim_{q'\to q}a_1(q,q')=-\lim_{q'\to q}(i\not{D})^2a_0(q,q')\\&=\lim_{q'\to q}\left\{\sigma_0g^{\alpha\beta}(D_\alpha D_\beta-\Gamma_{\alpha\beta}{}^\gamma D_\gamma)-\frac{1}{4}R\sigma_0+\frac{1}{2}e\epsilon^{\alpha\beta}F_{\alpha\beta}\sigma_3\right\}a_0(q,q')\\&=-\frac{1}{12}R\sigma_0+\frac{1}{2}e\epsilon^{\alpha\beta}F_{\alpha\beta}\sigma_3\,,\end{aligned} \tag{A·35}$$

where the conditions (A·32)–(A·34) have been used. Having obtained $a_0(q,q')$ and $a_1(q,q')$ in the limit $q' \to q$, the asymptotic form of $G(q,q',\tau)$ in this limit is immediately found from Eq. (A·27) to be

$$\lim_{q'\to q} G(q,q',\tau) = \frac{1}{4\pi\tau}\sigma_0 - \frac{1}{48\pi}R\sigma_0 + \frac{1}{8\pi}e\epsilon^{\alpha\beta}F_{\alpha\beta}\sigma_3 + O(\tau)\,, \qquad 0<\tau\ll 1\,. \tag{A·36}$$

Next, we insert Eq. (A·36) into the last line of Eq. (A·6). Then, the first and second terms on the right-hand side of Eq. (A·36) vanish, from Eq. (A·6), when the trace of the Pauli matrices is calculated. After taking the limit $\tau \searrow 0$, only the third term survives, and we have

$$\mathcal{A}_{\rm reg}(q) = \frac{e}{4\pi}{\rm tr}\,\epsilon^{\alpha\beta}F_{\alpha\beta}\,, \tag{A·37}$$

where the trace over the generators of $\mathcal{G}$ remains.

Let us next consider the chiral decomposition of the eigenfunction φ_N of the Dirac operator $i\not{D}$,[15),16)]

$$\varphi_N^{\pm} \equiv \frac{1}{2}(\sigma_0 \pm \sigma_3)\varphi_N\,, \quad \sigma_3\varphi_N^{\pm} = \pm\varphi_N^{\pm}\,. \tag{A·38}$$

The positive chirality component φ_N^+ and the negative chirality component $\varphi_{N'}^-$ are orthogonal in the sense that ${\varphi_N^+}^\dagger\varphi_{N'}^- = {\varphi_{N'}^-}^\dagger\varphi_N^+ = 0$. In terms of $\varphi_N^{\pm}$, due to the relation $\{i\not{D},\,\sigma_3\} = 0$, the eigenvalue equation (A·1) can be written as

$$i\not{D}\varphi_N^{\pm}(q) = \lambda_n\varphi_N^{\mp}(q)\,. \tag{A·39}$$

Now we assume that $\lambda_0 = 0$; accordingly, the corresponding eigenfunctions $\varphi_{0,\nu_\pm}^{\pm}$ are treated as the zero-modes of $i\not{D}$. Except for $\varphi_{0,\nu_\pm}^{\pm}$, the positive and negative chirality components are not eigenfunctions of $i\not{D}$. Equation (A·39) shows that when $n \neq 0$, or equivalently $\lambda_n \neq 0$, there is a one-to-one correspondence between φ_N^+ and φ_N^-. Consequently, it follows that the number of positive chirality components $\{\varphi_N^+\}_{n\neq 0}$ is equal to that of negative chirality components $\{\varphi_N^-\}_{n\neq 0}$. Also, when $n \neq 0$, the relation

$$\int_{\mathcal{M}} d^2q\sqrt{\mathfrak{g}(q)}\,{\varphi_N^+}^\dagger(q)\varphi_N^+(q) = \int_{\mathcal{M}} d^2q\sqrt{\mathfrak{g}(q)}\,{\varphi_N^-}^\dagger(q)\varphi_N^-(q) \tag{A·40}$$

can be proved using Eq. (A·39), together with the self-adjointness of $i\not{D}$ and the fact that λ_n is purely real. Since the zero-modes $\varphi_{0,\nu_\pm}^{\pm}$ are eigenfunctions of $i\not{D}$, the set of zero-modes $\{\varphi_{0,\nu_\pm}^{\pm}\}$ with the orthonormality condition

$$\int_{\mathcal{M}} d^2q\sqrt{\mathfrak{g}(q)}\,\varphi_{0,\nu_\pm}^{\pm\,\dagger}(q)\varphi_{0,\nu'_S}^{S}(q) = \begin{cases} \delta_{\nu_\pm\nu'_\pm}\,, & (S=\pm) \\ 0\,, & (S=\mp) \end{cases} \tag{A·41}$$

can be taken as a subset of the orthonormal set $\{\varphi_N\}$. In Eq. (A·41), the orthogonality relation in the case $S = \mp$ is valid owing to the condition $\varphi_{0,\nu_\pm}^{\pm\,\dagger}\varphi_{0,\nu'_\mp}^{\mp} = 0$, which is derived from the eigenvalue equation $\sigma_3\varphi_{0,\nu_\pm}^{\pm} = \pm\varphi_{0,\nu_\pm}^{\pm}$.

Using Eqs. (A·40) and (A·41), the integral of $\mathcal{A}_{\rm reg}(q)$ over $\mathcal{M}$ can be evaluated from the first line of Eq. (A·6), and we find

$$\begin{aligned}
&\int_{\mathcal{M}} d^2q\sqrt{\mathfrak{g}(q)}\,\mathcal{A}_{\rm reg}(q)\\
&=\lim_{\tau\searrow 0}\sum_N e^{-\tau\lambda_N^2}\int_{\mathcal{M}} d^2q\sqrt{\mathfrak{g}(q)}\,\big(\varphi_N^{+\dagger}(q)\varphi_N^{+}(q)-\varphi_N^{-\dagger}(q)\varphi_N^{-}(q)\big)\\
&=\sum_{\nu_+}\int_{\mathcal{M}} d^2q\sqrt{\mathfrak{g}(q)}\,\varphi_{0,\nu_+}^{+\dagger}(q)\varphi_{0,\nu_+}^{+}(q)-\sum_{\nu_-}\int_{\mathcal{M}} d^2q\sqrt{\mathfrak{g}(q)}\,\varphi_{0,\nu_-}^{-\dagger}(q)\varphi_{0,\nu_-}^{-}(q)\\
&=\mathfrak{n}_+-\mathfrak{n}_-\,,
\end{aligned}\tag{A·42}$$

where $\mathfrak{n}_+\equiv\sum_{\nu_+}1$ $\big(\mathfrak{n}_-\equiv\sum_{\nu_-}1\big)$ represents the number of positive (negative) chirality zero-modes. Combining Eqs. (A·37) and (A·42) and noting Eq. (A·19), we have

$$\mathfrak{n}_+-\mathfrak{n}_-=\frac{e}{4\pi}\int_{\mathcal{M}} d^2q\,\mathrm{tr}\,\varepsilon^{\alpha\beta}F_{\alpha\beta}\,.\tag{A·43}$$

Thus, the Atiyah-Singer index theorem in two dimensions, expressed by Eq. (2·6), is proven.

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