

## Proposing a New Constraint for Predictions of $pp$ , $\bar{p}p$ Total Cross Sections and $\rho$ Ratio at LHC

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We propose a new constraint(1) corresponding to the FESR (with the moment  $n = -1$ ) free from unphysical regions. Using this constraint(1) together with the constraint(2) (with the moment  $n = 1$ ), we search for the simultaneous best fit to the data points of  $\sigma_{\text{tot}}^{(+)}$  and  $\rho^{(+)}$  ratio up to the SPS energies to determine those values at higher energies. We then predict  $\sigma_{\text{tot}}^{(+)} = 107.1 \pm 2.6$  mb,  $\rho^{(+)} = 0.127 \pm 0.004$  at the LHC energy ( $\sqrt{s} = 14$  TeV).

Recently,<sup>1),2)</sup> we have searched for the simultaneous best fit of the average of  $pp$ ,  $\bar{p}p$  total cross sections ( $\sigma_{\text{tot}}^{(+)}$ ), and the ratio of the real to imaginary part of the forward scattering amplitude ( $\rho^{(+)}$ ) for  $70 \text{ GeV} < P_{\text{lab}} < P_{\text{SPS}}$  (up to the largest momentum of SPS corresponding to  $\sqrt{s} = 0.9$  TeV) in treating high-energy constants as parameters constrained by the finite-energy sum rule (FESR)<sup>5),6)</sup> with moment  $n = 1$ . We then predict  $\sigma_{\text{tot}}^{(+)}$  and  $\rho^{(+)}$  in the LHC ( $\sqrt{s} = 14$  TeV) as well as high-energy cosmic-ray regions. Block and Halzen<sup>3),4)</sup> have also reached the similar conclusions based on duality in a different approach.

### Proposal of a new constraint

The purpose of this paper is to propose the other new constraint besides the previous one in order to constrain the above constants. Following Ref. 1), we consider the crossing-even forward scattering amplitude defined by

$$F^{(+)}(\nu) = \frac{f^{\bar{p}p}(\nu) + f^{pp}(\nu)}{2} \quad \text{with} \quad \text{Im}F^{(+)}(\nu) = \frac{k\sigma_{\text{tot}}^{(+)}(\nu)}{4\pi}. \quad (1)$$

We also assume

$$\begin{aligned} \text{Im}F^{(+)}(\nu) &= \text{Im}R(\nu) + \text{Im}F_{P'}(\nu) \\ &= \frac{\nu}{M^2} \left( c_0 + c_1 \log \frac{\nu}{M} + c_2 \log^2 \frac{\nu}{M} \right) + \frac{\beta_{P'}}{M} \left( \frac{\nu}{M} \right)^{\alpha_{P'}} \end{aligned} \quad (2)$$

at high energies for  $\nu > N$ . Here,  $M$  is the proton (anti-proton) mass and  $\nu, k$  are the incident proton (anti-proton) energy, momentum in the laboratory system,

respectively. (We use the same notation as Refs. 1) and 2) in this article.) Defining<sup>\*</sup>

$$\tilde{F}^{(+)}(\nu) = F^{(+)}(\nu) - R(\nu) - F_{P'}(\nu) - F^{(+)}(0) \sim \nu^{\alpha(0)}, \quad (\alpha(0) < 0) \quad (3)$$

and write dispersion relation for  $\frac{\tilde{F}^{(+)}(\nu)}{\nu-M}$ . Since this amplitude is superconvergent, we have obtained<sup>1)</sup> in the spirit of  $P'$  sum rule<sup>8)</sup>

$$\begin{aligned} \text{Re}\tilde{F}^{(+)}(M) &= \frac{2P}{\pi} \int_0^M \frac{\nu}{k^2} \text{Im}F^{(+)}(\nu) d\nu + \frac{1}{2\pi^2} \int_0^{\bar{N}} \sigma_{\text{tot}}^{(+)}(k) dk \\ &\quad - \frac{2P}{\pi} \int_0^N \frac{\nu}{k^2} \left\{ \text{Im}R(\nu) + \frac{\beta_{P'}}{M} \left(\frac{\nu}{M}\right)^{0.5} \right\} d\nu, \end{aligned} \quad (4)$$

where  $\bar{N} = \sqrt{N^2 - M^2} \simeq N$ . The equation (4) was called FESR(1).<sup>\*\*)</sup> This FESR suffers from the unphysical regions coming from boson poles below the  $\bar{p}p$  threshold. Reliable estimates, however, are difficult. Therefore, we have not adopted the FESR(1) in the analysis.<sup>1),2)</sup>

Let us now change our strategy to use FESR(1), not at  $\nu = M$  but at some intermediate energy  $\nu = \nu_1$  as

$$\begin{aligned} \text{Re}\tilde{F}^{(+)}(\nu_1) &= \frac{2P}{\pi} \int_0^M \frac{\nu \text{Im}F^{(+)}(\nu)}{\nu^2 - \nu_1^2} d\nu + \frac{P}{2\pi^2} \int_M^N \frac{\nu k}{\nu^2 - \nu_1^2} \sigma_{\text{tot}}^{(+)}(\nu) d\nu \\ &\quad - \frac{2P}{\pi} \int_0^N \frac{\nu}{\nu^2 - \nu_1^2} \left\{ \text{Im}R(\nu) + \frac{\beta_{P'}}{M} \left(\frac{\nu}{M}\right)^{0.5} \right\} d\nu. \end{aligned} \quad (5)$$

If we choose the value of  $\nu_1$  to be sufficiently large, this constraint is not sensitive to the unphysical regions (the first term on the right-hand side of Eq. (5)) as well as ambiguities from low-energy integrals of  $pp$  and  $\bar{p}p$  scatterings.

Suppose we consider Eq. (5) with  $N = N_1$  and  $N = N_2$  ( $N_2 > N_1$ ). Taking the difference between these two relations, we obtain

$$\frac{2P}{\pi} \int_{N_1}^{N_2} \frac{\nu}{\nu^2 - \nu_1^2} \left\{ \text{Im}R(\nu) + \frac{\beta_{P'}}{M} \left(\frac{\nu}{M}\right)^{0.5} \right\} d\nu = \frac{P}{2\pi^2} \int_{N_1}^{N_2} \frac{\nu k}{\nu^2 - \nu_1^2} \sigma_{\text{tot}}^{(+)}(\nu) d\nu. \quad (6)$$

Let us call this relation as the constraint(1) which we use in our analysis. This constraint gives the relation between high-energy constants  $c_2, c_1, c_0, \beta_{P'}$ , and the cross-section integrals, and is free from the unphysical regions.

### The general approach

Besides the constraint(1), we have the FESR corresponding to  $n = 1$ ,<sup>6)</sup>

$$\begin{aligned} &\int_0^M \nu \text{Im} F^{(+)}(\nu) d\nu + \frac{1}{4\pi} \int_0^{\bar{N}} k^2 \sigma_{\text{tot}}^{(+)}(k) dk \\ &= \int_0^N \nu \text{Im} R(\nu) d\nu + \int_0^N \nu \text{Im} F_{P'}(\nu) d\nu. \end{aligned} \quad (7)$$

<sup>\*</sup>) Although  $\text{Re}F^{(+)}(\nu)$  becomes large for large values of  $\nu$ , a real constant has to be introduced in principle since the dispersion relation for  $\text{Re}F^{(+)}(\nu)$  requires a single subtraction constant  $F^{(+)}(0)$ .<sup>2),7)</sup>

<sup>\*\*)</sup> This FESR(1) corresponds to  $n = -1$ .<sup>6)</sup>

We call Eq. (7) as the constraint(2) which we also use in our analysis.

The  $\text{Re}F^{(+)}(\nu)$  is calculable from the  $\text{Im}F^{(+)}(\nu)$ , Eq. (2) by requiring the relation  $F^{(+)}(-\nu) = (F^{(+)}(\nu))^*$  to hold.<sup>9)</sup> Therefore, we obtain<sup>10)</sup>

$$\begin{aligned} \rho^{(+)}(\nu) &= \frac{\text{Re}F^{(+)}(\nu)}{\text{Im}F^{(+)}(\nu)} \\ &= \frac{\frac{\pi\nu}{2M^2} (c_1 + 2c_2 \log \frac{\nu}{M}) - \frac{\beta_{P'}}{M} (\frac{\nu}{M})^{0.5} + F^{(+)}(0)}{\frac{k\sigma_{\text{tot}}^{(+)}}{4\pi}}. \end{aligned} \quad (8)$$

The constraints(1), (2) and the formula of  $\sigma_{\text{tot}}^{(+)}$  (Eqs. (1) and (2)) and the  $\rho^{(+)}$  ratio (Eq. (8)) are our starting points. Among four constants,  $c_2, c_1, c_0$  and  $\beta_{P'}$ , the  $c_0$  and  $\beta_{P'}$  are represented by the other two ( $c_2$  and  $c_1$ ) by two constraints. Then, we search for the simultaneous best fit to  $\sigma_{\text{tot}}^{(+)}$  and  $\rho^{(+)}$  in treating three constants  $c_2, c_1$  and  $F^{(+)}(0)$  as parameters.

### Evaluation of cross-section integrals using experimental data

The integrals of  $\sigma_{\text{tot}}^{(+)}$  appearing on RHS (LHS) of Eq. (6) (Eq. (7)) are estimated by using experimental total cross sections  $\sigma_{\text{tot}}^{pp}$  and  $\sigma_{\text{tot}}^{\bar{p}p}$  in Particle Data Group 2004.<sup>11)</sup> A phenomenological formula,  $\text{Im}f^i(\nu) (= \frac{k}{4\pi} \sigma_{\text{tot}}^i) = \frac{\nu}{M} (c_0^i + c_1^i \log \frac{\nu}{M} + c_2^i \log^2 \frac{\nu}{M}) + \frac{\beta^i}{M} (\frac{\nu}{M})^{0.5} + \frac{d^i}{M} (\frac{\nu}{M})^{-0.5} + \frac{f^i}{M} (\frac{\nu}{M})^{-1.5}$  for  $i = \bar{p}p, pp$ , is used to fit experimental  $\sigma_{\text{tot}}^{\bar{p}p}$  ( $\sigma_{\text{tot}}^{pp}$ ) in the region of  $2.5 \text{ GeV} \leq k \leq 100 \text{ GeV}$  ( $2.592 \text{ GeV} \leq k \leq 100 \text{ GeV}$ ). The  $c_2^i$  and  $c_1^i$  are fixed with the values in our previous analysis,  $(c_2^i, c_1^i) = (0.0479, -0.186)$  (“analysis 2” in Ref. 2)), and the 77 (103) points of data are fitted with four parameters,  $c_0^i, \beta^i, d^i$  and  $f^i$ , respectively. The best-fitted values of  $\chi^2$  ( $\chi^2/(N_D - N_P)$ ) are  $148.5/(77-4)$  and  $71.8/(103-4)$ , respectively for  $\bar{p}p$  and  $pp$ . The large value of  $\chi^2$  for  $\bar{p}p$  comes from the inconsistency among the data of different experiments. In order to obtain good fit to  $\bar{p}p$  data, we must remove some outliers, points giving large  $\chi^2$  contributions. For this purpose we use statistical method, named Sieve algorithm.<sup>3), 12)</sup> In this method, by minimizing the Lorentzian squared,  $A_0^2 = \sum_{i=1}^{N_D} \ln\{1 + \gamma \Delta\chi_i^2\}$  (not the chi squared,  $\chi^2 = \sum_{i=1}^{N_D} \Delta\chi_i^2$ ) with  $\gamma = 0.179$ ,<sup>12)</sup> a “robust” fit is obtained to the same data, where  $N_D$  is the number of data points and  $\Delta\chi_i^2$  represents the  $\chi^2$ -contribution of the  $i$ -th point. Points giving  $\Delta\chi_i^2 > \Delta\chi_{\text{max}}^2$  in this robust fit are regarded as outliers, and removed. We take a cut  $\Delta\chi_{\text{max}}^2 = 6$ , and seven points are removed.<sup>\*)</sup> After removing these points, we obtain a sifted data set, re-fitted by minimizing the conventional  $\chi^2$ . As a result we obtain renormalized  $\chi^2$  (including the factor  $R = 1.140$ <sup>12)</sup>),  $\chi_{\bar{p}p}^2/(N_D - N_P) = 36.6/(70 - 4)$ . Finally we obtain the successful fits for both  $\sigma_{\text{tot}}^{\bar{p}p}$  and  $\sigma_{\text{tot}}^{pp}$ . The  $(c_0^i, \beta^i, d^i, f^i) = (6.34, 11.08, 5.28, -0.15)$  ( $(6.32, 4.25, -12.64, 24.4)$ ) are obtained for  $i = \bar{p}p$  ( $pp$ ).

We take the values of parameters appearing in Eq. (6) as  $(\bar{N}_1, \bar{N}_2, k_1) = (10, 70, 40)$  GeV, where  $\nu_1 = \sqrt{k_1^2 + M^2}$ . The above phenomenological fits give the cross-

<sup>\*)</sup>  $(k(\text{GeV}), \sigma_{\text{tot}}^{\bar{p}p}(\text{mb})) = (2.5, 79.4 \pm 1.0), (3.54, 69.7 \pm 0.5), (3.6, 76.2 \pm 1.8), (4., 71. \pm 1.), (4.015, 66.84 \pm 0.32), (4.3, 60.6 \pm 0.8), (9.14, 57.51 \pm 0.73)$  are removed.

section integrals  $\frac{P}{2\pi^2} \int_{\bar{N}_1}^{\bar{N}_2} \frac{\nu k}{\nu^2 - \nu_1^2} \sigma_{\text{tot}}^i(k) dk = 242.57 \pm 1.00$  ( $219.04 \pm 0.47$ )  $\text{GeV}^{-1}$  for  $i = \bar{p}p$  ( $pp$ ), where the errors correspond to the one-standard deviations. By averaging these values we obtain

$$\frac{P}{2\pi^2} \int_{\bar{N}_1}^{\bar{N}_2} \frac{\nu k}{\nu^2 - \nu_1^2} \sigma_{\text{tot}}^{(+)}(k) dk = 230.81 \pm 0.55 \text{ GeV}^{-1} . \quad (9)$$

We can also evaluate the cross-section integral in Eq. (7). We divide the region of integral into two parts,  $\frac{1}{4\pi} \int_0^{\bar{N}} k^2 \sigma_{\text{tot}}^i(k) dk = \frac{1}{4\pi} \int_0^{\bar{N}_0^i} k^2 \sigma_{\text{tot}}^i(k) dk + \frac{1}{4\pi} \int_{\bar{N}_0^i}^{\bar{N}} k^2 \sigma_{\text{tot}}^i(k) dk$ , and the integral in higher energy-region (the second term) is evaluated by using the phenomenological fit in the same manner. The integral in lower energy region (the first term) is evaluated by using experimental data directly: Each datum is connected with the next point by a straight line in order, and the resulting polygonal line graph gives the relevant integral. (The details of this procedure are explained in our previous works.<sup>1),2)</sup> By taking the  $\bar{N}$  as 10 GeV and  $\bar{N}_0^{\bar{p}p} = 4.7$  GeV, we obtain  $\frac{1}{4\pi} \int_0^{\bar{N}} k^2 \sigma_{\text{tot}}^{\bar{p}p}(k) dk = (524.62 \pm 2.89) + (3499.44 \pm 14.22) = 4024.06 \pm 14.51$  GeV for  $i = \bar{p}p$ . By taking  $\bar{N}_0^{pp} = 4.966$  GeV,  $\frac{1}{4\pi} \int_0^{\bar{N}} k^2 \sigma_{\text{tot}}^{pp}(k) dk = (357.24 \pm 0.89) + (2411.26 \pm 3.50) = 2768.50 \pm 3.61$  GeV for  $i = pp$ . By averaging them we obtain

$$\frac{1}{4\pi} \int_0^{\bar{N}} k^2 \sigma_{\text{tot}}^{(+)}(k) dk = 3396.3 \pm 7.5 \text{ GeV} \quad \text{with } \bar{N} = 10 \text{ GeV}. \quad (10)$$

This value is consistent with our previous estimate,  $3403 \pm 20$   $\text{GeV}$ ,<sup>1)</sup> which is evaluated by using the area of the polygonal line graphs up to  $k = \bar{N}$  ( $= 10$  GeV). In our present estimate, both of the integrals are estimated with small errors less than 0.3% .

### FESR as two constraints

By using the integrals, Eqs. (9) and (10), we obtain the constraints(1) and (2) as

$$\begin{aligned} \text{constraint(1)} & 3.316\beta_{P'} + 31.98c_0 + 141.1c_1 + 610.9c_2 = 230.81 , & (11) \\ & (\text{normalized } 0.104\beta_{P'} + c_0 + 4.41c_1 + 19.1c_2 = 7.22) , \end{aligned}$$

$$\begin{aligned} \text{constraint(2)} & 140.7\beta_{P'} + 383.6c_0 + 781.6c_1 + 1635.c_2 = 3396.3 , & (12) \\ & (\text{normalized } 0.367\beta_{P'} + c_0 + 2.04c_1 + 4.26c_2 = 8.85) , \end{aligned}$$

where we neglect the errors of cross-section integrals, and regard Eqs. (11) and (12) as exact constraints. The equations are also rewritten in the form with the coefficient of  $c_0$  normalized to unity in the parenthesis. Solving these two equations, we obtain the constraints for  $c_0$  and  $\beta_{P'}$  as

$$\begin{aligned} c_0 & = c_0(c_2, c_1) = 6.572 - 5.348c_1 - 24.95c_2, \\ \beta_{P'} & = \beta_{P'}(c_2, c_1) = 6.217 + 9.025c_1 + 56.40c_2 . \end{aligned} \quad (13)$$

### Data

We use the rich data of particle data group 2004.<sup>11)</sup> We take both statistical and systematic errors into account. The error of each datum  $\Delta y_i$  is taken to be  $\Delta y_i = \sqrt{(\Delta y_i)_{\text{stat}}^2 + (\Delta y_i)_{\text{syst.}}^2}$ .<sup>\*</sup> In case the data are listed in the same value of  $k$ , they are  $1/(\Delta y_i)^2$ -weighted averaged, respectively, for  $\sigma_{\text{tot}}^{\bar{p}p}$  and  $\sigma_{\text{tot}}^{pp}$ . Then, the data points with errors less than 3mb are selected, and when  $\sigma_{\text{tot}}^{\bar{p}p}$  and  $\sigma_{\text{tot}}^{pp}$  are listed at the same value of  $k$ , we can obtain  $\sigma_{\text{tot}}^{(+)}$  data points by averaging them. We obtain 27 points in  $10 \leq k \leq 2100$  GeV (up to ISR energy). The 5 points of  $\sigma_{\text{tot}}^{\bar{p}p}$  are reported in SPS energy  $1.5 \times 10^5 \leq k \leq 4.4 \times 10^5$  GeV. They are regarded as even-cross section  $\sigma_{\text{tot}}^{(+)}$ . Totally 32 points of  $\sigma_{\text{tot}}^{(+)}$  are obtained in  $10 \leq k \leq 4.4 \times 10^5$  GeV (up to SPS energy).

The  $\rho^{(+)}$  data points are similarly obtained by combining  $\rho^{\bar{p}p}$  and  $\rho^{pp}$  when they are listed at the same value of  $k$ .<sup>\*\*)</sup> We obtain 11 points in  $10 \leq k \leq 1500$  GeV. A  $\rho^{pp}$  point at  $k = 1.5597 \times 10^5$  GeV in SPS is regarded as a  $\rho^{(+)}$  datum. Totally we obtain 12 points of  $\rho^{(+)}$  in  $10 \leq k \leq 1.6 \times 10^5$  GeV.

### Analysis and result

In the actual analysis we fit the data of  $\text{Re}F^{(+)}(\nu)$  instead of  $\rho^{(+)}$ . The 32 points of  $\sigma_{\text{tot}}^{(+)}$  and 12 data points of  $\text{Re}F^{(+)}$  above  $k = 10$  GeV up to SPS energy  $\sqrt{s} = 0.9$  TeV are fitted simultaneously in treating the constants,  $c_2, c_1$  for  $\sigma_{\text{tot}}^{(+)}$  and  $c_2, c_1, F^{(+)}(0)$  for  $\text{Re}F^{(+)}$ , as parameters.

The results are shown in Fig. 1. The  $\chi^2/\text{d.o.f}$  is  $15.88/(44-3)$ , which is less than unity. The respective  $\chi^2$ -values divided by the number of data points for  $\sigma_{\text{tot}}^{(+)}$  and  $\rho^{(+)}$  are  $\chi_{\sigma}^2/N_{\sigma} = 10.50/32$  and  $\chi_{\rho}^2/N_{\rho} = 5.38/12$ , respectively. The fit is successful.

The estimated values of the constants are given in Table I. The result is compared with the previous analysis,<sup>2)</sup> where only Eq. (12) is used as a constraint.

In order to check our result, we take another value of  $k_1, k_1 = 80$  GeV for constraint(1) with the other parameters to remain unchanged ( $(\bar{N}_1, \bar{N}_2, \bar{N}) = (10, 70, 10)$  GeV). In this case the constraint(1) becomes  $-3.628\beta_{P'} - 27.68c_0 - 113.0c_1 - 463.3c_2 = -203.93 \pm 0.35$  GeV<sup>-1</sup> (normalized:  $0.131\beta_{P'} + c_0 + 4.08c_1 + 16.7c_2 = 7.37 \pm 0.01$ ). By using this together with the constraint(2) (Eq. (12)), the analysis is done in the same way. The value of  $c_2$  is obtained as  $c_2 = 0.0472 \pm 0.0050$  with  $\chi^2/\text{d.o.f} = 15.12/(44-3)$ . This  $c_2$  is almost the same as in Table I, and the result is considered to be almost independent of the value of  $k_1$ .

<sup>\*</sup>) In our previous papers,<sup>1),2)</sup> the treatment of systematic errors of  $\rho^{(+)}$  data is inadequate, although the results of analyses are affected only a little. By using corrected data of  $\rho^{(+)}$  explained below, the result of ‘‘analysis 2’’ in Ref. 2) is modified as follows: The best fitted value of  $c_2$  is  $c_2 = 0.0481 \pm 0.0042$  (previously  $0.0479 \pm 0.0037$ <sup>2)</sup>), and the predictions at LHC are  $\sigma_{\text{tot}}^{pp} = 107.6 \pm 2.8$  mb (previously  $107.8 \pm 2.4$  mb<sup>2)</sup>) and  $\rho^{pp} = 0.129 \pm 0.004$  (which remains the same as in Ref. 2)).

<sup>\*\*)</sup> The  $\rho^{pp}$  at  $k = 268.7$  GeV and  $\rho^{\bar{p}p}$  at  $k = 272$  GeV are combined into  $\rho^{(+)}$  at  $k = 270$  GeV. Three points of  $\rho^{pp}$  at  $k = 498$  GeV and a  $\rho^{\bar{p}p}$  at  $k = 491.5$  GeV are combined into  $\rho^{(+)}$  at  $k = 495$  GeV. The  $\rho^{pp}$  at  $k = 1484.7$  GeV and  $1490.3$  GeV and the  $\rho^{\bar{p}p}$  at  $k = 1473.5$  GeV are combined into  $\rho^{(+)}$  at  $k = 1480.5$  GeV. In this procedure the values of  $\sigma_{\text{tot}}^{\bar{p}p}$  and  $\sigma_{\text{tot}}^{pp}$  at respective  $k$ -values are necessary. They are obtained by the formula in Ref. 15).

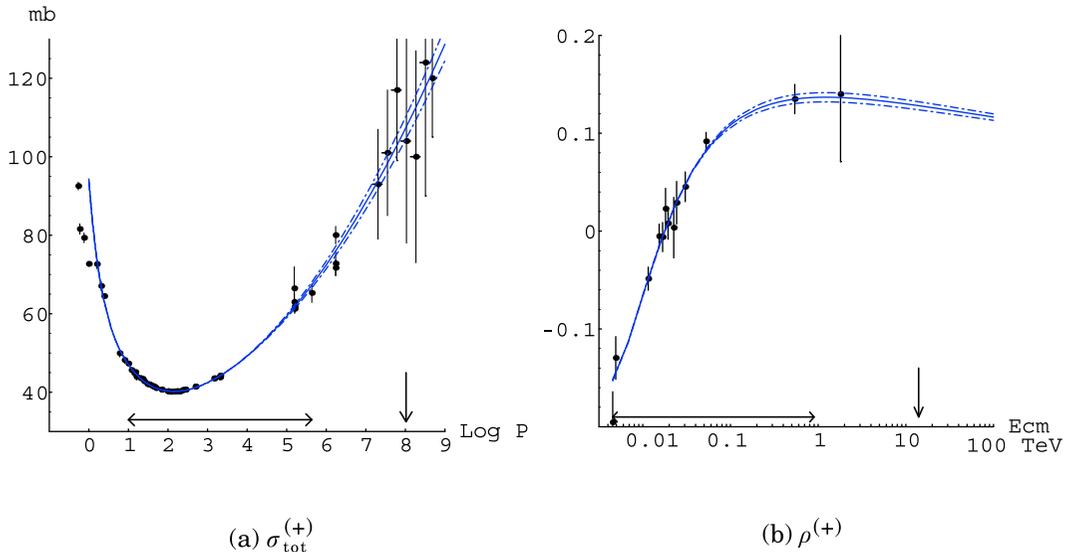


Fig. 1. Predictions for  $\sigma^{(+)}$  and  $\rho^{(+)}$ : (a) Total cross section  $\sigma_{\text{tot}}^{(+)}$  versus  $\log_{10} P_{\text{lab}} / \text{GeV}$ , (b) gives the  $\rho^{(+)}$  ( $= \text{Re } F^{(+)} / \text{Im } F^{(+)}$ ) versus  $E_{\text{cm}}$  in terms of TeV. The fit is done for the data up to the SPS energy, in the region  $10 \text{ GeV} \leq k \leq 4.3 \times 10^5 \text{ GeV}$  ( $4.54 \text{ GeV} \leq \sqrt{s} \leq 0.9 \text{ TeV}$ ), which is shown by horizontal arrow in each figure. The vertical arrow represents the LHC energy  $\sqrt{s} = 14 \text{ TeV}$ , corresponding to  $k = 1.04 \times 10^8 \text{ GeV}$ . The thin dot-dashed lines correspond to the one standard deviation of  $c_2$ , given with  $(c_2, c_1, c_0, \beta_{P'}, F^{(+)}(0)) = (0.0473 \pm 0.0042, -0.173 \mp 0.061, 6.32 \pm 0.22, 7.32 \mp 0.31, 10.01 \mp 0.28)$ .

Table I. Estimated values of the constants in the best fit using the constraint(1) with  $(\bar{N}_1, \bar{N}_2, k_1) = (10, 70, 40) \text{ GeV}$  and the constraint(2) with  $\bar{N} = 10 \text{ GeV}$ . The  $c_0$  and  $\beta_{P'}$  are represented by  $c_2$  and  $c_1$  through the constraints, and the fit is performed by using  $c_2$ ,  $c_1$  and  $F^{(+)}(0)$  as parameters, of which errors are given by the  $\chi^2$  function  $\chi^2(c_2, c_1, F^{(+)}(0))$ . Conversely by solving the constraints for  $c_2$  and  $c_1$ , the  $\chi^2$  function is represented by  $c_0$ ,  $\beta_{P'}$  and  $F^{(+)}(0)$ . The errors of  $c_0$  and  $\beta_{P'}$  are, thus, obtained.

| $c_2$               | $c_1$              | $c_0$           | $\beta_{P'}$    | $F^{(+)}(0)$     |
|---------------------|--------------------|-----------------|-----------------|------------------|
| $0.0473 \pm 0.0042$ | $-0.173 \pm 0.062$ | $6.32 \pm 0.23$ | $7.32 \pm 0.33$ | $10.01 \pm 1.79$ |

We have also checked the dependence of the errors of cross-section integrals, Eqs. (9) and (10). In the case when a larger value of the integral,  $230.81 + 0.55$  ( $3396.3 + 7.5$ ), is used for constraint(1) (constraint(2)) with the other integral to remain the same, we obtain that the best-fit value of  $c_2$  is  $0.0483$  ( $0.0465$ ) with  $\chi^2 = 14.68$  ( $15.04$ ). The deviations of  $c_2$  from the original value  $0.0473$  are  $0.0010$  ( $-0.0008$ ). They are small, compared with the statistical error  $0.0042$ : about  $\sqrt{10^2 + (-8)^2} / 42 = 30\%$ . So, we can regard Eqs. (11) and (12) as exact constraints.

Special attention has to be paid when the two constraints are employed to constrain the values of  $c_0$  and  $\beta_{P'}$ . As shown in Eqs. (11) and (12), the constraints(1) and (2) take the  $c_0$ -normalized forms  $0.104\beta_{P'} + c_0 + \dots$  and  $0.367\beta_{P'} + c_0 + \dots$ , respectively. They are linearly independent and we have obtained the meaningful result. If the parameters are badly taken so that two constraints are not sufficiently linearly

independent, the result becomes meaningless. For example, in case of  $k_1 = 60$  GeV, the constraint(1) has a normalized form  $0.314\beta_{P'} + c_0 + \dots$ . It is quite close to the constraint(2), and this selection of parameters is not suitable for the analysis. In the present analysis, in order to obtain sufficiently independent constraint, we have taken much larger value of  $\bar{N}_2$  ( $= 70$  GeV) for constraint(1) than  $\bar{N}$  ( $= 10$  GeV) for constraint(2).

It is pointed out<sup>4)</sup> that there are strong resemblances between our approach and the one by Block and Halzen.<sup>3)</sup> They estimated the values of experimental even-cross section  $\sigma_{\text{even}}(\nu_0)$  and of its derivative  $\frac{d\sigma_{\text{even}}}{d(\nu/M)}|_{\nu_0}$  at a certain energy  $\nu = \nu_0 = 7.59$  GeV ( $\sqrt{s} = 4$  GeV) by using a local fit. These two quantities are used as constraints<sup>13)</sup> for the constants  $c_2, c_1, c_0$  and  $\beta_{P'}$ , and  $c_0$  and  $\beta_{P'}$  are represented by  $c_2$  and  $c_1$ , similarly to our Eq. (13). They have shown in Ref. 4) that the constraint for  $\sigma_{\text{even}}(\nu_0)$ , which gives  $8.67 = c_0 + 2.091c_1 + 4.371c_2 + 0.3516\beta_{P'}^*$ , is very close to our FESR(2), Eq. (12). Numerical difference seems very small at a first look, but this difference is physically very important, since, in the former, the constraint is obtained at one point  $\nu = \nu_0$  in asymptotic region of  $\sigma_{\text{tot}}^{(+)}$ , while, in the latter, all the information in low-energy resonance region is included in the integral of  $\sigma_{\text{tot}}^{(+)}$  taken from  $k = 0$  to 10 GeV.

By using the values of constants in Table I, we can predict the  $\sigma_{\text{tot}}^{(+)}$  and  $\rho^{(+)}$  at Tevatron-collider energy ( $\sqrt{s} = 1.8$  TeV) and LHC energy ( $\sqrt{s} = 14$  TeV).

$$\begin{aligned} \sigma_{\text{tot}}^{(+)} &= 75.96 \pm 1.16 \text{ mb} \quad (\sqrt{s} = 1.8 \text{ TeV}), \quad 107.5 \pm 2.9 \text{ mb} \quad (\sqrt{s} = 14 \text{ TeV}), \\ \rho^{(+)} &= 0.136 \pm 0.005 \quad (\sqrt{s} = 1.8 \text{ TeV}), \quad 0.128 \pm 0.004 \quad (\sqrt{s} = 14 \text{ TeV}). \end{aligned} \quad (14)$$

The relevant energies are very high, and the difference between  $\sigma_{\text{tot}}^{\bar{p}p}(\nu)$  and  $\sigma_{\text{tot}}^{pp}(\nu)$  is expected to be very small:  $\frac{1}{2}(\sigma_{\text{tot}}^{\bar{p}p} - \sigma_{\text{tot}}^{pp}) \sim 0.003 \text{ mb}^{**}$  at LHC energy  $\sqrt{s} = 14$  TeV. Thus, the  $\sigma_{\text{tot}}^{(+)}$  and  $\rho^{(+)}$  in Eq. (14) can be regarded as  $\sigma_{\text{tot}}^{pp}$  and  $\rho^{pp}$ , which are directly compared with the coming LHC experiment.

Our predicted values are almost the same as the previous ones.<sup>2)</sup> They are consistent with the recent prediction by Block and Halzen<sup>3)</sup>  $\sigma_{\text{tot}}^{pp} = 75.19 \pm 0.55$  mb,  $\rho^{pp} = 0.139 \pm 0.001$  at Tevatron energy  $\sqrt{s} = 1.8$  TeV, and  $\sigma_{\text{tot}}^{pp} = 107.3 \pm 1.2$  mb,  $\rho^{pp} = 0.132 \pm 0.001$  at LHC energy  $\sqrt{s} = 14$  TeV. They also analyzed the crossing-odd amplitude. Their smaller errors compared with ours come from different treatment of the data. Our prediction has also to be compared with Cudell et al.<sup>15)</sup>  $\sigma_{\text{tot}}^{pp} = 111.5 \pm 1.2_{\text{sys}} \pm 4.1_{\text{stat}}$  mb,  $\rho^{pp} = 0.1361 \pm 0.0015_{\text{sys}} \pm 0.0058_{\text{stat}}$ , whose fitting techniques favour the CDF point at  $\sqrt{s} = 1.8$  TeV.

Finally we emphasize that our present analysis with two constraints is independent of the previous one<sup>1),2)</sup> with one constraint. Although the high-energy constants

<sup>\*</sup>) In Ref. 4), the constraint is given in units of mb and the LHS is given as 48.58 mb, which is replaced by 8.67 here in our notation where the constants  $c_i$  are dimensionless.

<sup>\*\*</sup>) This value is estimated by using the result of the analysis<sup>14)</sup> of crossing-odd amplitude,  $F^{(-)}(\nu) = (f^{\bar{p}p}(\nu) - f^{pp}(\nu))/2$ . It is represented by  $\text{Im}F^{(-)}(\nu) \simeq \frac{\beta_V}{M} \left(\frac{\nu}{M}\right)^{\alpha_V}$  through the exchange of the vector( $\rho, \omega$ )-meson trajectories. The values of constants,  $(\alpha_V, \beta_V) = (0.517, 3.64)$ , which are estimated through the best-fit to both  $\sigma_{\text{tot}}^{\bar{p}p}$  and  $\sigma_{\text{tot}}^{pp}$ , predict  $\frac{1}{2}(\sigma_{\text{tot}}^{\bar{p}p} - \sigma_{\text{tot}}^{pp}) = 0.0026$  mb at  $\sqrt{s} = 14$  TeV.

are strongly constrained by two FESR, Eqs. (11) and (12), in the present analysis, the result is almost the same with the previous one.<sup>2)</sup>

It is worthwhile to point out the following:

1. Both of the constants  $c_0$ ,  $\beta_{P'}$  are constrained as  $c_0 = c_0(c_2, c_1)$  and  $\beta_{P'} = \beta_{P'}(c_2, c_1)$  through FESR (namely duality).
2. And, then the high-energy behaviours of  $\sigma_{\text{tot}}^{(+)}$  have been  $\chi^2$  fitted in treating the constants  $c_2$ ,  $c_1$  as parameters since  $\sigma_{\text{tot}}^{(+)}$  is most sensitive to  $c_2$ .

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