

## Universal Scaling for the Jamming Transition

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The existence of universal scaling in the vicinity of the jamming transition of sheared granular materials is predicted by a phenomenological theory. The critical exponents are explicitly determined and are independent of the spatial dimension. The validity of the theory is verified by a molecular dynamics simulation.

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Jamming is an athermal phase transition between the solidlike jammed phase and the liquidlike unjammed phase of granular assemblies. Above the critical density, which is referred to as point J, the assemblies obtain rigidity and dynamic yield stress, while assemblies behave as dense liquids below point J. Liu and Nagel<sup>1)</sup> indicated that the jamming transition is a key concept of glassy materials. Since then, many similarities between the conventional glass transition and the jamming transition have been investigated.<sup>2)</sup> Indeed, there are many examples where granular materials are used in order to investigate dynamical heterogeneity in glassy materials.<sup>3)-6)</sup> On the other hand, we still do not have a unified viewpoint for describing glassy materials because we cannot use conventional theoretical tools to study the glass transition such as mode-coupling theory.<sup>7)</sup>

Jamming is a continuous transition that causes the bulk and shear moduli to become nonzero, and there are scaling laws in the vicinity of point J similar to those in conventional critical phenomena.<sup>8),9)</sup> Olsson and Teitel<sup>10)</sup> and Hatano<sup>11)</sup> further demonstrated the existence of elegant scaling laws near point J. Therefore, we can expect the existence of a simple theory for describing the jamming transition. However, we still do not have any theory to determine the critical exponents of the jamming transition.

In this paper, we predict the critical exponents for the jamming transition of sheared granular materials based on a phenomenological theory. First, we introduce the system under consideration and the scaling laws, some of which were introduced in Ref. 11). Second, on the basis of a phenomenological theory, we determine the critical exponents for the jamming transition. Finally, we verify the theoretical prediction by carrying out a simulation. The most surprising finding is that the

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critical exponents are independent of the spatial dimension  $D$  but depend on the type of particle interaction.

Let us consider a dense sheared and frictionless granular system in which uniform shear flow is stable. The system consists of  $N$  spherical grains in  $D$  dimensions. An important parameter used to characterize the system is the volume fraction  $\phi$ . In contrast to granular gases, the contact force plays a crucially important role in the jamming transition, where the normal contact force is repulsive and characterized by  $k\delta_{12}^\Delta$ . Here,  $k$  is the stiffness constant and  $\delta_{12} = r - (\sigma_1 + \sigma_2)/2$  where  $r$  is the relative distance between the particles in contact with diameters  $\sigma_1$  and  $\sigma_2$ . We consider that the Hertzian contact law  $\Delta = 3/2$  is appropriate for three-dimensional grains, but we often use a simpler linear spring model with  $\Delta = 1$ . In this paper, we disregard any tangential contact force between grains. Thus, granular particles are frictionless, which simplifies the argument.

As introduced in Ref. 11), this system is expected to exhibit scaling laws for the granular temperature  $T$  and shear stress  $S$  in the sheared plane near the jamming transition as

$$T = A_{T,D}|\Phi|^{x_\Phi}\mathcal{T}_\pm\left(t_D\frac{\dot{\gamma}}{|\Phi|^{x_\Phi/x_\gamma}}\right), \quad (1)$$

$$S = A_{S,D}|\Phi|^{y_\Phi}\mathcal{S}_\pm\left(s_D\frac{\dot{\gamma}}{|\Phi|^{y_\Phi/y_\gamma}}\right), \quad (2)$$

where  $\dot{\gamma}$  is the shear rate,  $\Phi \equiv \phi - \phi_J$  is the excess volume fraction compared with the critical fraction  $\phi_J$  at point J, and  $A_{T,D}$  and  $A_{S,D}$  are respectively the amplitudes of the granular temperature and shear stress. The scaling functions  $\mathcal{T}_+(x)$  and  $\mathcal{S}_+(x)$  above point J respectively differ from  $\mathcal{T}_-(x)$  and  $\mathcal{S}_-(x)$  below point J. It should be noted that  $A_{T,D}$ ,  $A_{S,D}$ ,  $t_D$  and  $s_D$  do not depend on  $\dot{\gamma}$  and  $\Phi$  but depend only on  $D$ .

We also introduce a characteristic time scale of sheared granular assemblies as

$$\omega \equiv \frac{\dot{\gamma}S}{nT}, \quad (3)$$

where  $n$  is the number density of grains. This  $\omega$  is reduced to the collision frequency in the unjammed phase in a steady state achieved by the balance between viscous heating and collisional energy loss. In contrast to the assumption by Hatano et al.,<sup>12)</sup>  $\omega$  also satisfies the scaling form

$$\omega = A_{w,D}|\Phi|^{z_\Phi}\mathcal{W}_\pm\left(w_D\frac{\dot{\gamma}}{|\Phi|^{z_\Phi/z_\gamma}}\right). \quad (4)$$

Thus, there are six critical exponents,  $x_\Phi$ ,  $x_\gamma$ ,  $y_\Phi$ ,  $y_\gamma$ ,  $z_\Phi$  and  $z_\gamma$ , in Eqs. (1), (2) and (4). We note that the normal stress  $P$  also satisfies a similar scaling relation,  $P = A_{p,D}|\Phi|^{y'_\Phi}\mathcal{P}_\pm(p_D\dot{\gamma}/|\Phi|^{y'_\Phi/y'_\gamma})$ ,<sup>11)</sup> but we omit the details of the arguments on  $y'_\Phi$  and  $y'_\gamma$  in this paper. We will discuss them elsewhere.

Below point J, Bagnold's scaling should hold.<sup>13)-15)</sup> Thus, the scaling functions in the unjammed branch satisfy

$$\lim_{x \rightarrow 0} \mathcal{T}_-(x) = \lim_{x \rightarrow 0} \mathcal{S}_-(x) = x^2, \quad \lim_{x \rightarrow 0} \mathcal{W}_-(x) = x. \quad (5)$$

On the other hand, the jammed branch is characterized by the dynamic yield stress and the freezing of motion. Then, the scaling functions in the jammed branch satisfy

$$\lim_{x \rightarrow 0} \mathcal{S}_+(x) = \lim_{x \rightarrow 0} \mathcal{W}_+(x) = 1, \quad \lim_{x \rightarrow 0} \mathcal{T}_+(x) = x. \tag{6}$$

To obtain the last equation in Eq. (6), we have used Eq. (3) and the other two equations in Eq. (6). Since the scaling functions are independent of  $\Phi$  at point J, we obtain

$$\lim_{x \rightarrow \infty} \mathcal{T}_\pm(x) \propto x^{x_\gamma}, \quad \lim_{x \rightarrow \infty} \mathcal{S}_\pm(x) \propto x^{y_\gamma}, \quad \lim_{x \rightarrow \infty} \mathcal{W}_\pm(x) \propto x^{z_\gamma}. \tag{7}$$

We now determine the six critical exponents. First, we note that there are three trivial relations among the exponents. From Eqs. (1)–(4) and (6) we obtain

$$z_\Phi = y_\Phi - x_\Phi(1 - x_\gamma^{-1}). \tag{8}$$

Similarly, from Eqs. (1)–(4) with Eq. (5) or (7) we respectively obtain

$$z_\Phi(1 - z_\gamma^{-1}) = y_\Phi(1 - 2y_\gamma^{-1}) - x_\Phi(1 - 2x_\gamma^{-1}), \tag{9}$$

$$z_\gamma = y_\gamma - x_\gamma + 1. \tag{10}$$

Thus, we need three further relations to determine the exponents.

In order to obtain the other relations, we consider the pressure  $P$  in the limit  $\dot{\gamma} \rightarrow 0$ . Let us consider Cauchy’s stress in the jammed phase in which the pressure  $P$  is given by  $P = \sum_{i>j} \langle F_{ij}r_{ij} \rangle / V$ , where  $V$  is the volume of the system, and  $r_{ij}$  and  $F_{ij}$  are respectively the distance and force between particles  $i$  and  $j$ . This expression may be approximated by  $P \simeq Z(\Phi)r_c(\Phi)F_c(\Phi)$  in the zero shear limit, where  $Z(\Phi)$  is the average coordination number, and  $r_c(\Phi)$  and  $F_c(\Phi)$  are respectively the average distance between grains in contact and the average force acting on the contact point. It is obvious that  $Z(\Phi)$  and  $r_c(\Phi)$  can be replaced by  $Z(0)$  and  $r_c(0) = \sigma$  in the vicinity of the jamming point, where  $\sigma$  is the average diameter of the particles. Indeed, O’Hern et al.<sup>9)</sup> verified that  $Z(\Phi) - Z(0) \propto \Phi^{1/2}$  for three dimensional cases. Thus, the most important term is the mean contact force  $F_c(\Phi) \propto \delta(\Phi)^\Delta$ , where  $\delta(\Phi)$  is the average length of compression. Now, let us compress the system at the critical point  $\phi_J$  into  $\Phi = \phi - \phi_J > 0$  by an affine transformation. Since all the characteristic lengths are scaled by the system size, we may assume the approximate relation  $r_c(\Phi) = (\phi_J/\phi)^{1/D}\sigma$ . From the relation  $\delta(\Phi) = r_c(0) - r_c(\Phi)$ ,  $\delta(\Phi)$  approximately satisfies  $\delta(\Phi) \simeq (\sigma/D\phi_J)\Phi \sim \Phi$  in the vicinity of  $\Phi = 0$ . Thus, we conclude that  $P \sim \Phi^\Delta$ . This relation has also been verified in Ref. 9).

On the other hand, it is well known that Coulomb’s frictional law applies in granular systems in which  $S/P$  is a constant. Indeed, Hatano<sup>16)</sup> simulated the sheared granular system under a constant pressure  $P$  and demonstrated that the ratio satisfies  $\lim_{\dot{\gamma} \rightarrow 0} S(\dot{\gamma}, P)/P = S_Y(P)/P = M_0$ , where  $S_Y(P) \equiv \lim_{\dot{\gamma} \rightarrow 0} S(\dot{\gamma}, P)$  and the constant  $M_0$  is independent of the pressure  $P$ . The excess volume fraction  $\Phi(\dot{\gamma}, P)$  in this system is a function of  $\dot{\gamma}$  and  $P$ , but we can also express the pressure as  $P(\dot{\gamma}, \Phi)$ . Since  $M_0 = \lim_{\dot{\gamma} \rightarrow 0} S(\dot{\gamma}, P(\dot{\gamma}, \Phi))/P(\dot{\gamma}, \Phi) = S_Y(P(0, \Phi))/P(0, \Phi)$  is independent of

$P(0, \Phi)$ ,  $M_0$  should be independent of  $\Phi$ . Thus, we can conclude that the  $\Phi$  dependence of  $S$  is the same as that of  $P$  in the limit  $\dot{\gamma} \rightarrow 0$ . From this result, we obtain  $y_\Phi = \Delta$  and

$$y_\Phi = 1 \quad \text{for the linear spring model.} \quad (11)$$

This result also implies that  $y_\Phi = y'_\Phi$  which is consistent with the numerical observation.<sup>11)</sup>

We next consider the density of states. Wyart et al.<sup>17)</sup> demonstrated the following for unsheared assemblies of elastic soft spheres. (i) Jamming is related to the appearance of soft modes in the density of states. (ii) There is a plateau in the density of states in the vicinity of the jamming transition. (iii) The cutoff frequency  $\omega^*$  of the plateau is proportional to  $\sqrt{P}$ . From the argument in the previous paragraph, the applied pressure satisfies the relation  $P \propto \Phi^\Delta$ . When we assume that the characteristic frequency  $\omega$  in the limit  $\dot{\gamma} \rightarrow 0$  can be scaled by the cutoff frequency  $\omega^*$ , we may conclude that  $\omega \sim |\Phi|^{1/2}$  for the linear spring model. Thus, we obtain  $z_\Phi = \Delta/2$  or

$$z_\Phi = 1/2 \quad \text{for the linear spring model.} \quad (12)$$

Finally, we consider the characteristic frequency  $\omega$  in the unjammed phase ( $\Phi < 0$ ). In this phase, the characteristic frequency  $\omega$  is estimated as  $\omega \sim \sqrt{T/m/l(\Phi)}$ , where  $l(\Phi)$  is the mean free path. Note that  $l(\Phi)$  may be evaluated as  $(\sigma/D\phi_J)|\Phi|$  in the vicinity of point J using a parallel argument to  $\delta(\Phi)$  for  $\Phi > 0$ . From the scalings in Bagnold's regime (5), we obtain  $\omega \sim |\Phi|^{z_\Phi(1-z_\gamma^{-1})\dot{\gamma}}$  and  $T \sim |\Phi|^{x_\Phi(1-2x_\gamma^{-1})\dot{\gamma}^2}$ . Substituting these relations into  $\omega \sim \sqrt{T/m/l(\Phi)}$ , we obtain

$$z_\Phi(1 - z_\gamma^{-1}) - \frac{1}{2}x_\Phi(1 - 2x_\gamma^{-1}) = -1. \quad (13)$$

From the above six relations (8)–(13) we finally determine the six critical exponents as

$$\begin{aligned} x_\Phi &= 3, & x_\gamma &= \frac{6}{5}, & y_\Phi &= 1, \\ y_\gamma &= \frac{2}{5}, & z_\Phi &= \frac{1}{2}, & z_\gamma &= \frac{1}{5} \end{aligned} \quad (14)$$

for the linear spring model. The exponents in the Hertzian model are, of course, different. In the general situation for  $\Delta$ , Eq. (14) is replaced by

$$\begin{aligned} x_\Phi &= 2 + \Delta, & x_\gamma &= \frac{2\Delta + 4}{\Delta + 4}, & y_\Phi &= \Delta, \\ y_\gamma &= \frac{2\Delta}{\Delta + 4}, & z_\Phi &= \frac{\Delta}{2}, & z_\gamma &= \frac{\Delta}{\Delta + 4}. \end{aligned} \quad (15)$$

Note that the exponents are independent of the spatial dimension. This is not surprising because our phenomenological theory used to derive Eqs. (11) and (12) is independent of the spatial dimension.<sup>9),17)</sup> We stress an interesting feature of the jamming transition; the exponents strongly depend on the interaction model among particles. This property is in contrast to that in conventional critical phenomena.

Thus, we should be careful when using the idea of universality in describing the jamming transition.

Hereafter, we verify our theoretical results by a molecular dynamics simulation. In our simulation, the system consists of  $N$  spherical grains in 2, 3 and 4 dimensions. We adopt the linear spring model ( $\Delta = 1$ ) for simplicity. We also introduce a dissipative force  $-\eta\delta v$ , where  $\delta v$  represents the relative velocity between the particles in contact. Each grain has an identical mass  $m$ . In order to realize uniform velocity gradient  $\dot{\gamma}$  in the  $y$  direction and macroscopic velocity only in the  $x$  direction, we adopt the Lees-Edwards boundary conditions. The particle diameters are  $0.7\sigma_0$ ,  $0.8\sigma_0$ ,  $0.9\sigma_0$  and  $\sigma_0$ , each of which is assigned to  $N/4$  particles.

In our simulation  $m$ ,  $\sigma_0$  and  $\eta$  are set to unity, and all quantities are converted to dimensionless forms, where the unit of the time scale is  $m/\eta$ . We use the spring constant  $k = 1.0$ . For systems near the critical density, such as  $\phi = 0.8428$  for  $D = 2$ , and  $\phi = 0.643$  and  $0.6443$  for  $D = 3$ , we use  $N = 4000$  in order to remove finite-size effects. For other systems, we use  $N = 2000$ .

The scaling plots of our simulation based on the exponents in Eq. (14) are shown in Fig. 1. We stress that these scaling plots contain the data for  $D = 2, 3$  and  $4$ . The volume fraction at point J is estimated as  $\phi_J = 0.84285$  for  $D = 2$ ,  $\phi_J = 0.64455$  for  $D = 3$  and  $\phi_J = 0.4615$  for  $D = 4$ . The shear rate  $\dot{\gamma}$  is in the range between  $5 \times 10^{-7}$  and  $5 \times 10^{-5}$  for  $D = 2$  and  $3$  and between  $5 \times 10^{-6}$  and  $5 \times 10^{-4}$  for  $D = 4$ . The amplitudes and the adjustable parameters are obtained as  $(t_D, A_{t,D}, s_D, A_{s,D}, w_D, A_{w,D}) = (0.0125, 7.17, 0.025, 0.035, 0.05, 0.3)$  for  $D = 2$ ,  $(0.01385, 2.527, 0.03, 0.04, 0.065, 0.65)$  for  $D = 3$ , and  $(0.015, 1.6275, 0.03, 0.06, 0.06, 1)$  for  $D = 4$ . Since Fig. 1 exhibits elegant scaling laws, our phenomenological method seems to be correct.

Figure 2(a) shows the  $|\Phi|$  dependence of the shear viscosity  $\mu \equiv S/\dot{\gamma}$  in Bagnold's regime. Note that there is no consensus in previous studies on the shear viscosity. For example, Garcia-Rojo et al.<sup>18)</sup> reported that  $\mu \sim 1/(\phi_c - \phi)$ , where  $\phi_c$  is lower than  $\phi_J$ , while Losert et al.<sup>19)</sup> observed an exponent larger than 1 from their experiment. Also, an exponent of divergence in Ref. 10) is between 1 and 2. We also note that the viscosity is believed to diverge as  $|\Phi|^{-2}$  for colloidal suspensions.<sup>20)</sup> However, our scaling theory predicts  $\mu \propto |\Phi|^{y_\phi(1-2/y_\gamma)} \propto |\Phi|^{-4}$ , and the viscosity diverges at point J. Here, the scaling exponent for  $\mu$  is independent of  $\Delta$  because  $y_\phi$  and  $y_\gamma$  for an arbitrary  $\Delta$  are determined from our theory as in Eq. (15). As can be seen in Fig. 2(a), the theoretical prediction is consistent with our numerical result. We also examine the possibility that the viscosity diverges when  $\phi_c < \phi_J$  with  $\mu \sim (\phi_c - \phi)^{-1}$ , as in the case of Garcia-Rojo et al.<sup>18)</sup> Actually we can fit the data of our two-dimensional simulation by  $\mu \sim (\phi_c - \phi)^{-1}$  with  $\phi_c = 0.835$ , which is less than  $\phi_J = 0.8428$  for  $\phi < \phi_c$ , but the viscosity is still finite even for  $\phi > \phi_c$  (see Fig. 2(b)). Thus, we can conclude that (i) the viscosity does not satisfy  $(\phi_c - \phi)^{-1}$  but exhibits consistent behavior with  $(\phi_J - \phi)^{-4}$ , as predicted by our phenomenological theory, and (ii) the critical behaviors are only characterized by point J.

We also verify the validity of  $\omega \sim |\Phi|^{1/2}$  in the jammed phase from our simulation in Fig. 3. The envelope line of our result seems to be consistent with the theoretical prediction.

It should be noted that the existence of the plateau for  $|\Phi| \rightarrow 0$  in Fig. 2(a) can

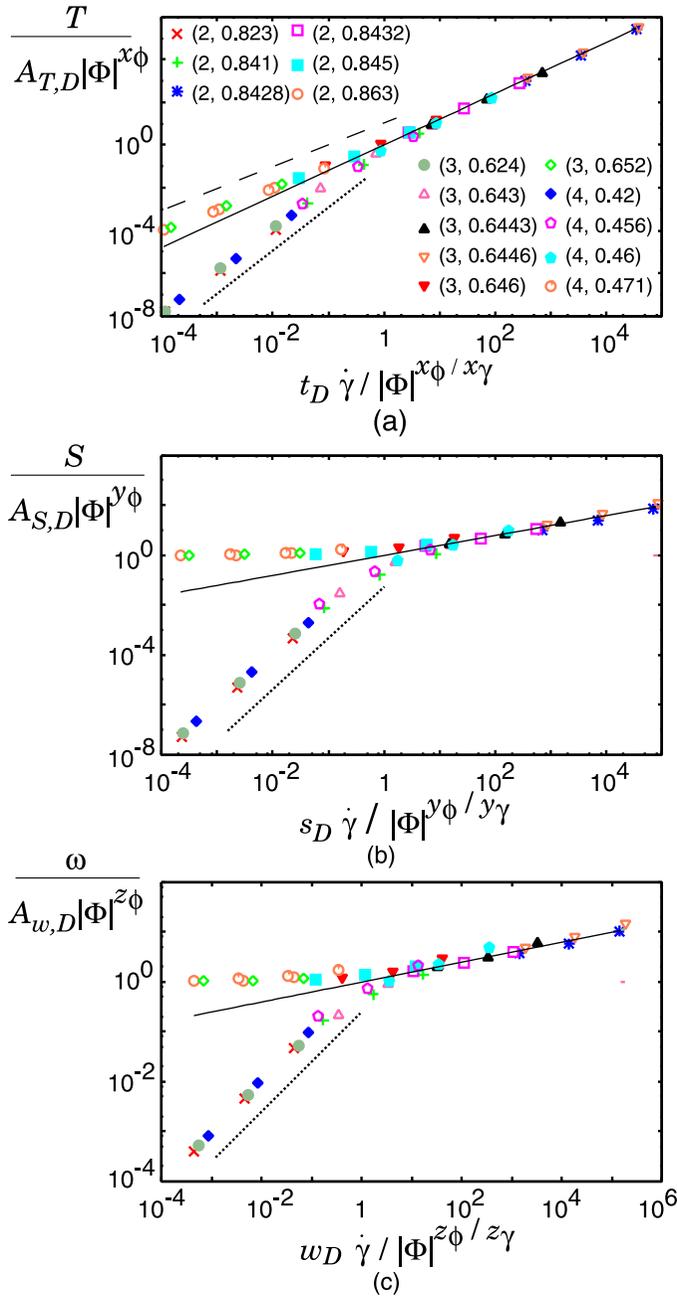


Fig. 1. (Color online) (a) Collapsed data of the shear rate dependence of the granular temperature  $T$  using the scaling law for  $D = 2, 3$  and  $4$ . The dashed line, dotted line and solid line are proportional to  $\dot{\gamma}$ ,  $\dot{\gamma}^2$  and  $\dot{\gamma}^{x\gamma}$ , respectively. The legends show the dimension  $D$  and volume fraction  $\phi$  as  $(D, \phi)$ . (b) Collapsed data of the shear rate dependence of the shear stress  $S$  using the scaling law for  $D = 2, 3$  and  $4$ . The dotted line and solid line are proportional to  $\dot{\gamma}^2$  and  $\dot{\gamma}^{y\gamma}$ , respectively. (c) Collapsed data of the shear rate dependence of the cooling rate  $\omega$  using the scaling law for  $D = 2, 3$  and  $4$ . The dotted line and solid line are proportional to  $\dot{\gamma}^2$  and  $\dot{\gamma}^{z\gamma}$ , respectively.

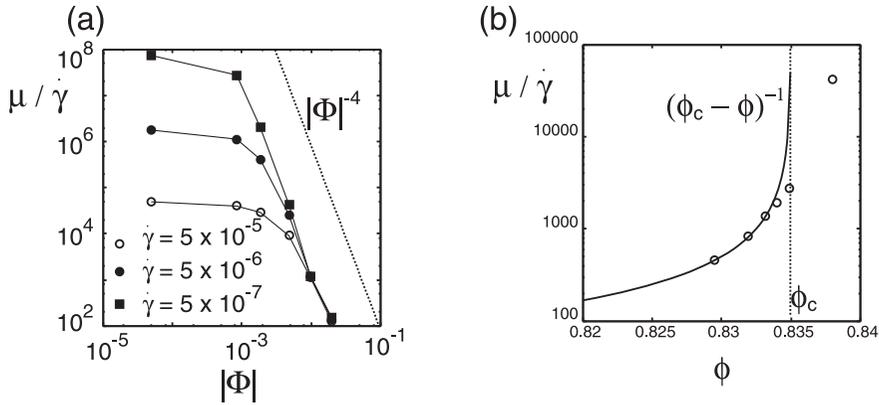


Fig. 2. (a)  $\mu/\dot{\gamma}$  as a function of  $\Phi$  for  $D = 2$  with  $\dot{\gamma} = 5 \times 10^{-5}$ ,  $5 \times 10^{-6}$  and  $5 \times 10^{-7}$  in the unjammed phase. (b)  $\mu/\dot{\gamma}$  as a function of  $\phi$  for  $\dot{\gamma} = 5 \times 10^{-7}$  in the unjammed phase, where the solid line is proportional to  $(\phi_c - \phi)^{-1}$  with  $\phi_c = 0.835$ .

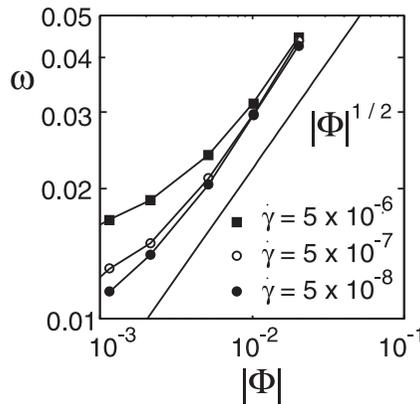


Fig. 3.  $\omega$  as a function of  $\Phi$  for  $D = 2$  with  $\dot{\gamma} = 5 \times 10^{-6}$ ,  $5 \times 10^{-7}$  and  $5 \times 10^{-8}$  in the jammed phase.

be understood from the scaling relation (2). Indeed, the  $\Phi$  dependence of the shear stress disappears in the limit of large  $x \equiv \dot{\gamma}/|\Phi|^{y_\phi/y_\gamma}$  with  $\mathcal{S}_-(x) \rightarrow x^{y_\gamma}$  as indicated by Eq. (7). Thus, we obtain  $\mu/\dot{\gamma} = S/\dot{\gamma}^2 \sim \dot{\gamma}^{y_\gamma-2} \sim \dot{\gamma}^{-8/5}$ . This estimation might be consistent with the simulation in Fig. 2(a), in which the value of the plateau increases as the shear rate decreases. Similarly, we can expect the height of the plateau of  $\omega$  to be  $\dot{\gamma}^{z_\gamma}$ , although this saturation cannot be verified from Fig. 3.

We now discuss our results. First of all, the ratios between the exponents  $x_\phi/x_\gamma$ ,  $y_\phi/y_\gamma$  and  $z_\phi/z_\gamma$  obtained by Eq. (14) or (15) satisfy

$$\alpha \equiv \frac{x_\phi}{x_\gamma} = \frac{y_\phi}{y_\gamma} = \frac{z_\phi}{z_\gamma} = \frac{\Delta + 4}{2}. \tag{16}$$

This is not surprising because the time scale is expected to be scaled by the shear rate. Thus, the ratio  $\alpha$  in Eqs. (1), (2) and (4) should be common. In other words, the characteristic time scale  $\tau$  exhibits critical slowing down as  $\tau \sim |\Phi|^{-\alpha}$ . This

property has already been indicated by Hatano.<sup>11)</sup> Once we accept the ansatz (16), Eqs. (8)–(10) are degenerate and reduce to

$$x_\phi - y_\phi + z_\phi = \alpha. \quad (17)$$

Equation (13) is also reduced to the simplified form

$$z_\phi = \frac{x_\phi}{2} - 1. \quad (18)$$

From these equations and Eqs. (11) and (12) with (16), we obtain Eq. (14) or (15).

Second, Hatano estimated the exponents to be  $x_\phi = 2.5$ ,  $x_\gamma = 1.3$ ,  $y_\phi = 1.2$  and  $y_\gamma = 0.57$  from his three-dimensional simulation based on the linear spring model,<sup>11)</sup> which differ from our prediction (14). In particular, if we use these values with Eq. (18),  $z_\phi$  is estimated as  $z_\phi = 0.25$ , which is one-half of our prediction of  $z_\phi = 1/2$ . However, the estimated scaling exponents strongly depend on the choice of  $\phi_J$  and the range of the shear rate  $\dot{\gamma}$ . The value of  $\phi_J$  and the range of  $\dot{\gamma}$  in Ref. 11) are larger than ours. If we adopt Hatano's  $\phi_J$  and the range of  $\dot{\gamma}$ , our numerical data can be scaled by Hatano's scaling. Although his scaling can be used over a wide range of  $\dot{\gamma}$ , deviation from his scaling can be detected in the small- $\dot{\gamma}$  region ( $\dot{\gamma} < 10^{-4}$ ). It is obvious that we should use as small  $\dot{\gamma}$  as possible to extract the critical properties. This suggests that our exponents are more appropriate than Hatano's exponents in characterizing the jamming transition. The difficulty in determining the exponents from the simulation also supports the significance of our theory for determining the scaling laws.

Third, the exponents obviously depend on the model of interaction between particles as predicted in Eq. (15). Our preliminary simulation suggests that the numerical exponents for the Hertzian contact model are consistent with the prediction of Eq. (15). Numerical results for the  $\Delta$  dependence of the exponents will be reported elsewhere.

Fourth, our results should be modified when we analyze the model in the zero temperature limit of a Langevin thermostat. This situation corresponds to that in Ref. 10). In this case, we should replace Bagnold's law in the unjammed phase with Newtonian law  $S \propto \dot{\gamma}$ . As a result, all the scaling exponents have different values. We will discuss the results of this situation elsewhere.

Finally, we comment on the relation between our results and previous studies on dynamical heterogeneity in glassy materials. The dynamical heterogeneity in glassy materials is characterized by large fluctuations of the four-point correlation function, in which the result strongly depends on the spatial dimension. On the other hand, our theory and numerical simulation suggest that the critical fluctuation is not important and that our phenomenological theory works well. Since the quantities we analyzed in this paper are not directly related to the four-point correlation function, there is no distinct contradiction between them. To study the roles of critical fluctuations and dynamical heterogeneity we may need a more sophisticated theory. This will be our future task.

In conclusion, we develop a phenomenological theory for describing the jamming transition. We determine the critical exponents, which are independent of the spatial

dimension. The validity of our theory has been verified by a molecular dynamics simulation.

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