

Overlooked Branch Cut in Steepest Descent Method

— *Switching Line and Atomic Domain* —

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One type of branch cut has been overlooked in the steepest descent method. This type of branch cut, which we call the *switching line*, is inevitable in this method if one wants to obtain the asymptotic evaluation of a target integral $\mathcal{I}(\mathbf{x})$ globally for any value of control parameter \mathbf{x} . The appearance of switching line on the space of control parameter can be viewed as one example of Berry's adiabatic phase. The recognition of the switching line leads us to yield a useful recipe of steepest descent method. Within the recipe, the whole space of control parameter is divided into multiple domains by the three types of boundary line (saddle connection line, ordinary branch cut and switching line). Each resultant domain obtained from this division is termed the *atomic domain*. The topological feature of stationary phase paths over each atomic domain can be reduced to what we call the *connectivity graph*. With use of the connectivity graph, the final asymptotic expression of a given integral $\mathcal{I}(\mathbf{x})$ for any value of control parameter \mathbf{x} can be presented very clearly without any ambiguity.

Subject Index: 013, 064

§1. Introduction

The *steepest descent method* is an excellent method^{1)–8)} with a long history to evaluate asymptotically a given integral of the form

$$\mathcal{I}(C, \mathbf{x}) = \int_C a(s; \mathbf{x}) e^{-f(s; \mathbf{x})/\epsilon} ds. \quad (1.1)$$

Here, $f(s; \mathbf{x})$ and $a(s; \mathbf{x})$ are some complex analytic functions of the *state variable* $s \in \mathbb{C}$ and the *control parameters* $\mathbf{x} \in \mathbb{C}^K$ ($K \in \mathbb{N}$); ϵ is a positive small parameter to represent the asymptotic limit; C is an integration path that connects a valley of integrand to another valley of it. In principle, the steepest descent method consists of the following two steps.

The first step is to deform the integration path C into a sequence of *stationary phase paths* $\{E^{(l)}(\mathbf{x}) : l = 0, 1, \dots, L-1\}$ without changing the value of integral $\mathcal{I}(C, \mathbf{x})$ according to Cauchy's integral theorem. Each stationary phase path

$E^{(l)}(\mathbf{x})$ is a path along which the exponential factor $e^{-f(s;\mathbf{x})/\epsilon}$ in the integrand $a(s;\mathbf{x})e^{-f(s;\mathbf{x})/\epsilon}$ takes a constant phase. By this deformation of path, the given integral $\mathcal{I}(C, \mathbf{x})$ is decomposed into a set of integrals for stationary phase paths as

$$\mathcal{I}(C, \mathbf{x}) = \sum_{l=0}^{L-1} \mathcal{I}(E^{(l)}(\mathbf{x}), \mathbf{x}) . \quad (1.2)$$

We refer to this analysis how the integration path C is deformed into a sequence of stationary phase paths as the *global analysis* in steepest descent method.

The second step is the *local analysis*. With use of the Gauss integral, the integral $\mathcal{I}(E^{(l)}(\mathbf{x}), \mathbf{x})$ for each stationary phase path $E^{(l)}(\mathbf{x})$ is evaluated as

$$\mathcal{I}(E^{(l)}(\mathbf{x}), \mathbf{x}) \sim \sqrt{\frac{2\pi\epsilon}{f''(s_{\text{II}}^{(l)}(\mathbf{x}); \mathbf{x})}} a(s_{\text{II}}^{(l)}(\mathbf{x}); \mathbf{x}) e^{-f(s_{\text{II}}^{(l)}(\mathbf{x}); \mathbf{x})/\epsilon} , \quad (1.3)$$

where $s_{\text{II}}^{(l)}(\mathbf{x})$ is the *saddle point*^{*}) on the stationary phase path $E^{(l)}(\mathbf{x})$. Throughout this paper, the prime ($'$) denotes the differentiation with respect to s .

The combination of the global analytic formula (1.2) and the local analytic formula (1.3) yields the asymptotic evaluation

$$\mathcal{I}_{\text{II}}(C, \mathbf{x}) = \sum_{l=0}^{L-1} \sqrt{\frac{2\pi\epsilon}{f''(s_{\text{II}}^{(l)}(\mathbf{x}); \mathbf{x})}} a(s_{\text{II}}^{(l)}(\mathbf{x}); \mathbf{x}) e^{-f(s_{\text{II}}^{(l)}(\mathbf{x}); \mathbf{x})/\epsilon} \quad (1.4)$$

of the given integral $\mathcal{I}(C, \mathbf{x})$. As is well-known,⁶⁾ if a stationary phase path $E^{(l)}(\mathbf{x})$ experiences the *saddle connection phenomenon*^{**)} at some non-generic value of control parameter \mathbf{x} , the asymptotic evaluation $\mathcal{I}_{\text{II}}(C, \mathbf{x})$ (1.4) can change discontinuously there.

On the other hand, it is not so emphasized that the mathematical *expression* of the asymptotic evaluation $\mathcal{I}_{\text{II}}(C, \mathbf{x})$ experiences some *superficial change* when the control parameter crosses the branch cut^{***)} or what we call the *switching line*. The switching line is a sort of branch cut arising from the double-valuedness brought from the square root function $\sqrt{\frac{2\pi\epsilon}{f''(s_{\text{II}}(\mathbf{x}); \mathbf{x})}}$ in the local asymptotic evaluation (1.3). It is also found that the appearance of switching line on the control parameter space $\{\mathbf{x} \in \mathbb{C}^K\}$ can be viewed as one example of Berry's adiabatic phase.^{14), 15), †)}

^{*}) The suffix II for saddle point s_{II} comes from a correspondence between the traditional *steepest descent method* and the traditional *WKB method*.^{9), 10)} This correspondence is explained in our subsequent paper.¹¹⁾ The traditional WKB method for the Schrödinger equation is based on a quadratic equation as explained in Refs. 9) and 10) and accordingly many mathematical symbols that appear in them for the traditional WKB method have the suffix II. We here follow this convention to name the saddle point in the traditional steepest descent method.

^{**)} We distinguish the saddle connection phenomenon and the *Stokes phenomenon*^{12), 13)} by following the convention of Ref. 6) (see also §3.8).

^{***)} The branch cut is necessary on the control parameter space $\{\mathbf{x} \in \mathbb{C}^K\}$ to make the saddle point $s_{\text{II}}(\mathbf{x})$ be a single-valued function of the control parameter \mathbf{x} .

^{†)} This viewpoint is suggested by Sinji Iida (Ryukoku University).

The above two types of branch cut may be less important than the saddle connection line, at which a saddle connection phenomenon occurs. However, for anyone who wants to obtain correctly the asymptotic evaluation $\mathcal{I}_{\text{II}}(C, \mathbf{x})$ of a target integral $\mathcal{I}(C, \mathbf{x})$, these branch cuts are inevitable. Actually, just the definition of switching line can yield a principle how to choose the branch of square root function $\sqrt{\cdots}$ in (1.3). If this branch is treated incorrectly, a spurious discontinuity as a function of \mathbf{x} remains wrongly in the final asymptotic evaluation $\mathcal{I}_{\text{II}}(C, \mathbf{x})$.

The recognition of the switching line leads us to yield a useful recipe of steepest descent method. Within the recipe, the whole space of control parameter is divided into multiple domains by the three types of boundary line (saddle connection line, ordinary branch cut and switching line). Each resultant domain obtained from this division is termed the *atomic domain*. The topological feature of stationary phase paths over each atomic domain (e.g. Fig. 11 in §5) can be reduced to what we call the *connectivity graph* (e.g. Fig. 12).

This paper is organized as follows. In §2, some basic notions in steepest descent method are summarized. In §3, we explain in detail what is the switching line and why it is inevitable in steepest descent method through the application to the Airy function $\text{Ai}(x)$,¹⁶⁾ which is the fold diffraction integral.¹⁷⁾ Based on the knowledge of the switching line, §4 presents a recipe of steepest descent method. With use of this recipe, the asymptotic evaluation of the BNR integral^{18),19),*)} is derived very smoothly in §5.

Originally, our study in the present paper was started in order to prepare a firm basis for our another study to seek a new steepest descent method, which may be called the *divergence-free steepest descent method*.^{**)} The recipe of steepest descent method presented in this paper will be used heavily in our subsequent papers^{11),21)} as an essential tool to describe the divergence-free steepest descent method.

§2. Asymptotic formulae in steepest descent method

In this section 2, we summarize the asymptotic formulae in steepest descent method. We will see that the terms in graph theory^{22),23),***)} are very useful to express these asymptotic formulae.

2.1. Valley V and hill H

Our target integral for asymptotic analysis is

*) The BNR integral is well-known as an integral that exhibits the *new Stokes phenomenon*^{18),20)} and can be a nice example to utilize our recipe of steepest descent method.

***) Our study to seek the divergence-free steepest descent method for integrals is intimately related to our previous study of divergence-free WKB method^{9),10)} for differential equations.

***) One of the authors (S. A.) and Takeshi Yanagisawa (Tokyo Institute of Technology and Apple Computer, Inc.) have developed a convenient framework to express the graph theory in order to use it in a grammatical analysis of dynamical symbol sequences.²³⁾ The graph theory used in the present paper is fully based on this framework. We thank T. Y. to give us the permission to use freely the framework to express the graph theory in our paper.

$$\mathcal{I}(C) = \int_C e^{-f(s)/\epsilon} ds . \quad (2.1)$$

The analytic function $f(s)$ may have some hidden control parameters $\mathbf{x} \in \mathbb{C}^K$ ($K \in \mathbb{N}$); these control parameters are considered to be fixed until we proceed to investigate bifurcation phenomena on \mathbf{x} . In addition, the amplitude factor $a(s)$ is disposed from the integrand.*)

The integrand $e^{-f(s)/\epsilon}$ is supposed to have multiple valleys in the complex s -plane $\{s \in \mathbb{C}\}$. Here, each valley V indicates a sector of arguments of integration variable s over which the magnitude of integrand $|e^{-f(s)/\epsilon}|$ attenuates when the absolute value $|s|$ of s tends to the infinity, i.e.

$$V = \{s \in \mathbb{C} : |e^{-f(s)/\epsilon}| \rightarrow 0 \text{ as } |s| \rightarrow \infty\} . \quad (2.2)$$

On the other hand, the sector of arguments of s over which the magnitude of integrand $|e^{-f(s)/\epsilon}|$ diverges when $|s| \rightarrow \infty$ is called the *hill* and is denoted by H , i.e.

$$H = \{s \in \mathbb{C} : |e^{-f(s)/\epsilon}| \rightarrow \infty \text{ as } |s| \rightarrow \infty\} . \quad (2.3)$$

In the integral (2.1), the path C is supposed to run from a valley to another valley. The former valley is termed the *initial valley* ∂^-C of path C , whereas the latter valley is termed the *terminal valley* ∂^+C of it.

We consider a deformation of the original path C to another path C' while the initial and terminal valleys are invariant. In addition, the region enclosed by two paths C and C' is supposed to have no singularity of the integrand $e^{-f(s)/\epsilon}$. Then, such a deformation of path from C to C' does not change the value of integral $\mathcal{I}(C)$ according to Cauchy's integral theorem. This means that the value of integral $\mathcal{I}(C)$ depends on the path C only through its initial valley ∂^-C and its terminal valley ∂^+C . Accordingly, the integral $\mathcal{I}(C)$ for a path C is rewritten as

$$\mathcal{I}(C) = \mathcal{I}(\partial^-C, \partial^+C) , \quad (2.4)$$

where the integral $\mathcal{I}(V_i, V_j)$ for two arbitrary valleys V_i and V_j is defined by

$$\mathcal{I}(V_i, V_j) = \int_{V_i \rightarrow V_j} e^{-f(s)/\epsilon} ds . \quad (2.5)$$

Here $V_i \rightarrow V_j$ represents an arbitrary path from V_i to V_j .

2.2. Stationary phase path E and saddle point s_{II}

2.2.1. Stationary phase path E

Next, the integral $\mathcal{I}(V_i, V_j)$ (2.5) for two valleys V_i and V_j needs to be evaluated accurately by some asymptotic method. It is remembered that the path $V_i \rightarrow V_j$ in (2.5) is arbitrary as long as it connects the initial valley V_i to the terminal valley V_j .

*) In the asymptotic limit of $\epsilon \rightarrow 0$, the amplitude factor $a(s)$ is insignificant compared with the exponential factor $e^{-f(s)/\epsilon}$. The presence of the amplitude factor $a(s)$ just changes the local asymptotic evaluation (see §2.7).

One can utilize this freedom to choose a path in order to yield an accurate evaluation of integral $\mathcal{I}(V_i, V_j)$.

According to the traditional theory of steepest descent, a special path E on which the integrand $e^{-f(s)/\epsilon}$ has a constant phase is chosen as the path $V_i \rightarrow V_j^*$ because the contributions from small line segments along this path E to the integral $\mathcal{I}(V_i, V_j)$ do not cancel each other. Any point s on this path E satisfies

$$\arg e^{-f(s)/\epsilon} = \text{const} \quad \text{for any } s \in E . \quad (2.6)$$

Such a path E is called the *stationary phase path*. We here introduce the notation $V_i \xrightarrow{E} V_j$ for two valleys V_i, V_j and a stationary phase path E to represent the case that the stationary phase path E connects the valley V_i to the valley V_j , i.e.

$$V_i \xrightarrow{E} V_j \stackrel{\text{def}}{\iff} \partial^- E = V_i \wedge \partial^+ E = V_j . \quad (2.7)$$

We next consider the inversion of a stationary phase path E . A stationary phase path E connects a valley V_i to a valley V_j , if and only if the inverted path $-E$, which is also a stationary phase path, connects inversely the valley V_j to the valley V_i , i.e.

$$V_i \xrightarrow{E} V_j \iff V_i \xleftarrow{-E} V_j . \quad (2.8)$$

For two given valleys V_i and V_j , if some stationary phase path E connects the valley V_i to the valley V_j or equivalently if its inverted stationary phase path $-E$ connects the valley V_j to the valley V_i , then these valleys are said to be *directly connected*.

When we want to evaluate asymptotically an integral $\mathcal{I}(V_i, V_j)$ for two valleys V_i and V_j that are directly connected, this asymptotic evaluation is obtained by a local analysis that utilizes the stationary phase path E connecting V_i to V_j . On the other hand, if the initial valley V_i and the terminal valley V_j are *not* directly connected, we need an additional preparation for the asymptotic evaluation of integral $\mathcal{I}(V_i, V_j)$. This preparation is explained later in §2.4.

2.2.2. Steepest ascent line, steepest descent line and saddle point

According to the Cauchy-Riemann relation for analytic functions, any stationary phase path E of the integrand $e^{-f(s)/\epsilon}$ is parallel or anti-parallel to the gradient direction of the magnitude of integrand $|e^{-f(s)/\epsilon}|$ at each point s on E . Whether a path E is parallel or anti-parallel to the gradient direction of $|e^{-f(s)/\epsilon}|$ can be switched only at a point $s = s_{\text{II}}$ where the derivative of the analytic function $f(s)$ vanishes. This point s_{II} , which satisfies

$$(\text{SE}) : \quad f'(s_{\text{II}}) = 0 , \quad (2.9)$$

is the *saddle point* of the integrand $e^{-f(s)/\epsilon}$. Any stationary phase path E of the integrand $e^{-f(s)/\epsilon}$ that runs from a valley V_i to another valley V_j has always at least one saddle point s_{II} on it.

*) We note that such path E connecting directly V_i to V_j does not always exist.

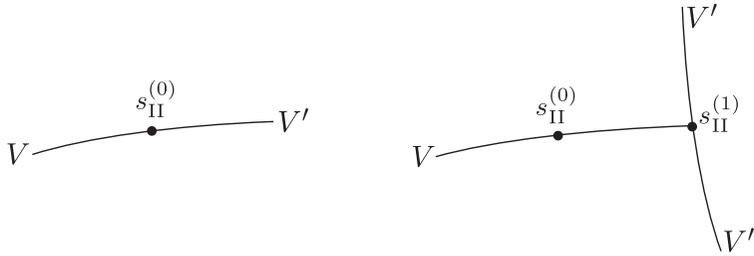


Fig. 1. **(Left)** No saddle connection (co-dimension zero). **(Right)** Saddle connection (co-dimension one). The solid curve indicates a stationary phase path E that runs from a valley to another valley. The dots (\bullet) on the path E are the saddle points $s_{II}^{(k)}$ ($k = 0, 1$) of the integrand $e^{-f(s)/\epsilon}$. The end points of the path E are the valleys, where its magnitude $|e^{-f(s)/\epsilon}|$ attenuates.

2.2.3. Saddle connection

We here cease hiding the control parameter $\mathbf{x} \in \mathbb{C}^K$ from the given integral $\mathcal{I}(C)$ (2.1) in order to explain a bifurcation phenomenon, which is called the *saddle connection*.⁶⁾

When the control parameter \mathbf{x} takes a generic value in the space of control parameter, each stationary phase path E calculated for this value of control parameter \mathbf{x} has just one saddle point s_{II} on it. By varying the value of control parameter \mathbf{x} from such a generic value and calculating the stationary phase paths repeatedly, it is possible to find some special value of control parameter \mathbf{x} for which a stationary phase path E has multiple saddle points $s_{II}^{(0)}, s_{II}^{(1)}, \dots, s_{II}^{(k)}$ ($1 \leq k \leq K$) on it. The event in which a stationary phase path E has $(k + 1)$ saddle points is co-dimension $k \geq 1$. This non-generic event, which is called the *saddle connection*,⁶⁾ is observed later in §§3 and 5 (see also Fig. 1).

2.3. Name of stationary phase paths and name of saddle points

Subsequently in this section 2, we consider only the generic case for the control parameter \mathbf{x} , at which no saddle connection occurs. Then, each stationary phase path E has the unique saddle point s_{II} on it, whereas its inverted stationary phase path $-E$ also has the same saddle point s_{II} on it. We use this fact to introduce a simple convention to name the stationary phase paths.

For explanation, we assume that the saddle point equation $f'(s_{II}) = 0$ is an M -th order algebraic equation with respect to s_{II} and the roots of this algebraic equation are named $\{s_{II,m} : m = 0, 1, \dots, M - 1\}$. For each saddle point $s_{II,m}$, there are two stationary phase paths that pass through the saddle point $s_{II,m}$ in opposite directions, since the generic case is considered. In this situation, our convention to name the stationary phase paths is as follows:

The two stationary phase paths that pass through a saddle point $s_{II,m}$ in opposite directions are named E_m and $-E_m$.

The stationary phase path E_m ($-E_m$) is termed the *positive (negative) stationary phase path* and is said to trace the *positive (negative) direction*.

It is further necessary to determine which stationary phase path is named E_m between two stationary phase paths that pass through a saddle point $s_{II,m}$. Our con-

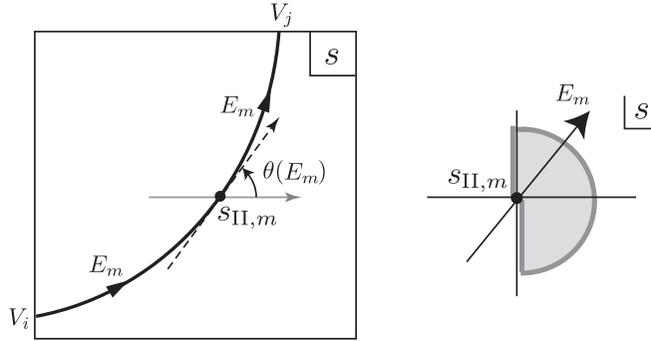


Fig. 2. Definition of rotation angle $\theta(E_m)$ for a stationary phase path E_m . This angle is measured from the positive $\Re s$ direction (gray arrow) to the tangential direction (dashed arrow) of E_m at $s = s_{II,m}$ in anticlockwise rotation. In the right figure, the shaded half circle denotes the range of rotation angle $-\frac{\pi}{2} < \theta(E_m) \leq \frac{\pi}{2}$ which any *positive* stationary phase path E_m belongs to.

vention to specify the positive stationary phase path E_m and the negative stationary phase path $-E_m$ is as follows:

- The tangential direction of any *positive* stationary phase path E_m at $s_{II,m} \in E_m$ points to the *right* half-plane of complex s -plane $\{s \in \mathbb{C}\}$.
- On the other hand, the tangential direction of any *negative* stationary phase path $-E_m$ at $s_{II,m} \in -E_m$ points to the *left* half-plane of it.

To simply express our convention of the positive direction for stationary phase paths, we introduce a geometrical angle on the complex s -plane $\{s \in \mathbb{C}\}$, which is called the *rotation angle*. The rotation angle $\theta(\pm E_m)$ of a stationary phase path $\pm E_m$ denotes the geometrical angle from the positive $\Re s$ direction to the tangential direction of $\pm E_m$ at the saddle point $s = s_{II,m}$ in anticlockwise rotation (see Fig. 2). This rotation angle $\theta(\pm E_m)$ has an ambiguity of integer multiples of 2π .

By using the rotation angle $\theta(\pm E_m)$ of a stationary phase path $\pm E_m$, our convention of the positive direction for stationary phase path is simply expressed as

$$-\frac{\pi}{2} < \theta(E_m) \leq \frac{\pi}{2} \quad \text{and} \quad \frac{\pi}{2} < \theta(-E_m) \leq \frac{3\pi}{2} \quad \text{mod } 2\pi . \quad (2.10)$$

In this convention, the *positive* imaginary axis is included in the *right* half-plane of complex s -plane $\{s \in \mathbb{C}\}$, whereas the *negative* imaginary axis is included in the *left* half-plane of it.

Subsequently, the integral $\mathcal{I}(E_m) = \int_{E_m} e^{-f(s)/\epsilon} ds$ for any positive stationary phase path E_m is termed the *elementary integral*.

2.4. Decomposition formula for the integral $\mathcal{I}(V_i, V_j)$

As mentioned in §2.2.1, when the initial valley V_i and the terminal valley V_j are directly connected, the asymptotic evaluation of integral $\mathcal{I}(V_i, V_j)$ for these valleys is obtained by a local analysis that utilizes the stationary phase path connecting V_i to V_j . On the other hand, in order to obtain the asymptotic evaluation of integral $\mathcal{I}(V_i, V_j)$ for two valleys V_i and V_j that are *not* directly connected, the integral $\mathcal{I}(V_i, V_j)$ needs to be decomposed into multiple elementary integrals $\{\mathcal{I}(E_m)\}$. Such

a decomposition is explained below.

2.4.1. Decomposition of integral $\mathcal{I}(V_i, V_j)$ into multiple elementary integrals

For two given valleys V_i and V_j , which may not be directly connected, we consider an alternating sequence

$$\gamma = (V^{(0)}, \sigma^{(0)}, E^{(0)}, V^{(1)}, \sigma^{(1)}, E^{(1)}, \dots, \sigma^{(L-1)}, E^{(L-1)}, V^{(L)}) \tag{2.11}$$

of valleys $\{V^{(l)} : l = 0, 1, \dots, L\}$ and positive stationary phase paths $\{E^{(l)} : l = 0, 1, \dots, L - 1\}$ with signs $\{\sigma^{(l)} : l = 0, 1, \dots, L - 1\}$ that satisfy the following conditions:

- (i) if an l -th sign $\sigma^{(l)}$ is positive, the l -th stationary phase path $E^{(l)}$ connects the l -th valley $V^{(l)}$ to the $(l + 1)$ -th valley $V^{(l+1)}$;
- (ii) otherwise if the l -th sign $\sigma^{(l)}$ is negative, the l -th stationary phase path $E^{(l)}$ connects the $(l + 1)$ -th valley $V^{(l+1)}$ to the l -th valley $V^{(l)}$;
- (iii) the initial valley $V^{(0)}$ and the terminal valley $V^{(L)}$ of the sequence γ (2.11) are respectively V_i and V_j .

The conditions (i) and (ii) mean that the direct connectivity

$$\begin{cases} V^{(l)} \xrightarrow{E^{(l)}} V^{(l+1)} & \sigma^{(l)} = +1 \\ V^{(l)} \xleftarrow{E^{(l)}} V^{(l+1)} & \sigma^{(l)} = -1 \end{cases} \quad \text{for } l = 0, 1, \dots, L - 1 \tag{2.12}$$

holds. The last condition (iii) means that the boundary condition

$$V^{(0)} = V_i \quad \text{and} \quad V^{(L)} = V_j \tag{2.13}$$

holds.

Next, according to Cauchy’s integral theorem, if there exists a sequence γ (2.11) from the valley V_i to the valley V_j , then the integral $\mathcal{I}(V_i, V_j)$ for these valleys can be expressed in terms of multiple elementary integrals as

$$\mathcal{I}(V_i, V_j) = \sum_{l=0}^{L-1} \sigma^{(l)} \mathcal{I}(E^{(l)}) . \tag{2.14}$$

2.4.2. An example of the decomposition of integral $\mathcal{I}(V_i, V_j)$

We offer to the reader an example of the decomposition of integral $\mathcal{I}(V_i, V_j)$ by using the configuration depicted in Fig. 3. This is taken from the application to the BNR integral,^{18),19)} which is later explained in §5.

In this situation, we let the initial valley and the terminal valley be V_0 and V_3 respectively. Then, the sequence γ (2.11) from V_0 to V_3 is represented as

$$\gamma = (V_0, -1, E_1, V_1, -1, E_2, V_2, +1, E_0, V_3) . \tag{2.15}$$

Next, the integral $\mathcal{I}(V_0, V_3)$ for these valleys V_0 and V_3 is decomposed into elementary integrals as

$$\mathcal{I}(V_0, V_3) = -\mathcal{I}(E_1) - \mathcal{I}(E_2) + \mathcal{I}(E_0) . \tag{2.16}$$

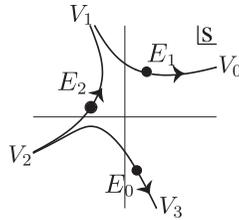


Fig. 3. Configuration of valleys $\{V_n : n = 0, 1, 2, 3\}$ and positive stationary phase paths $\{E_m : m = 0, 1, 2\}$ on the complex s -plane. The arrow (\rightarrow) attached to each saddle point (dot \bullet) represents the direction of the positive stationary phase path E_m .

2.5. Integrals defined for connectivity graph

This section 2.5 explains a systematic way to write down the above decomposition (2.14) of a given integral $\mathcal{I}(V_i, V_j)$ by using what we call the *connectivity graph*.

2.5.1. Introduction of connectivity graph

The connectivity graph G is introduced by regarding each valley V as a *vertex*^{*)} and each stationary phase path E as an *edge* in the graph G .^{**)} Any edge corresponding to each positive (negative) stationary phase path E_m ($-E_m$) is named a *positive (negative) edge*. Furthermore, if a positive stationary phase path E_m connects directly a valley V to a valley V' , then the initial vertex $\partial^- E_m$ and the terminal vertex $\partial^+ E_m$ of the edge E_m are the corresponding vertices V and V' respectively.

We let \mathbf{V} and \mathbf{E} denote respectively the whole set of vertices and the whole set of positive edges. Furthermore, the two functions $\partial^-, \partial^+ : \mathbf{E} \rightarrow \mathbf{V}$ from the set of edges \mathbf{E} to the set of vertices \mathbf{V} are called the *incidence relations*. Then, the connectivity graph G for a given integrand $e^{-f(s)/\epsilon}$ is formally defined by 4-tuple as

$$G = (\mathbf{V}, \mathbf{E}, \partial^-, \partial^+) . \tag{2.17}$$

2.5.2. Integral defined for vertices and integral defined for edges

Up to now, we have introduced various kinds of integral. At the beginning, we were given an integral $\mathcal{I}(C)$ (2.1) defined for a path C on the complex s -plane $\{s \in \mathbb{C}\}$. We next introduce another integral $\mathcal{I}(V_i, V_j)$ (2.5) defined for valleys V_i and V_j .

On the other hand, we have introduced in this section 2.5 the connectivity graph G for a given integrand $e^{-f(s)/\epsilon}$. In this description, each valley V of integrand $e^{-f(s)/\epsilon}$ corresponds in a one-to-one manner to a vertex $V \in \mathbf{V}$, whereas each positive stationary phase path E_m of integrand $e^{-f(s)/\epsilon}$ corresponds in a one-to-one manner to a positive edge $E_m \in \mathbf{E}$.

By using these one-to-one correspondences, we now introduce an integral $\mathcal{I}(V_i, V_j)$ for two *vertices* $V_i, V_j \in \mathbf{V}$ and another integral $\mathcal{I}(E_m)$ for a positive *edge* $E_m \in \mathbf{E}$.

^{*)} When a valley V is regarded as the corresponding vertex, this vertex is also denoted by V . Similarly, when a stationary phase path E is regarded as the corresponding edge, this edge is also denoted by E .

^{**)} In this paper, the directed graph is called simply the *graph*. Similarly, the directional edge is called the *edge*.

The integral $\mathcal{I}(V_i, V_j)$ for two vertices $V_i, V_j \in \mathbf{V}$ is defined to be equal to the integral $\mathcal{I}(V_i, V_j)$ for the corresponding valleys V_i and V_j . The integral $\mathcal{I}(E_m)$ for a positive edge $E_m \in \mathbf{E}$ is defined to be equal to the integral $\mathcal{I}(E_m)$ for the corresponding positive stationary phase path E_m .

2.5.3. Integral defined for paths in graph theory

In the previous section 2.4, we have considered a decomposition of the integral $\mathcal{I}(V_i, V_j)$ from an initial valley V_i to a terminal valley V_j into the summation $\sum_{l=0}^{L-1} \sigma^{(l)} \mathcal{I}(E^{(l)})$ (2.14) of multiple elementary integrals. This is achieved by finding a sequence γ (2.11) of valleys $\{V^{(l)}\}$ and positive stationary phase paths $\{E^{(l)}\}$ with signs $\{\sigma^{(l)}\}$ satisfying the direct connectivity (2.12) and the boundary condition (2.13).

We now reinterpret this decomposition formula $\mathcal{I}(V_i, V_j) = \sum \sigma^{(l)} \mathcal{I}(E^{(l)})$ (2.14) in terms of the connectivity graph \mathbf{G} (2.17). First, the direct connectivity (2.12) and the boundary condition (2.13) imply that the sequence γ (2.11) represents a *path**) in the connectivity graph \mathbf{G} that connects the initial vertex $V_i \in \mathbf{V}$ to the terminal vertex $V_j \in \mathbf{V}$, i.e.**) .

$$\gamma \in \Gamma(\mathbf{G}; V_i, V_j) . \quad (2.18)$$

Second, the decomposition formula (2.14) implies that the integral $\mathcal{I}(V_i, V_j)$ for vertices $V_i, V_j \in \mathbf{V}$ is expressed by the function $\mathcal{I}(\gamma)$ for a path γ (2.18), which can be naturally induced from the integral $\mathcal{I}(E_m)$ for an edge $E_m \in \mathbf{E}$, i.e.

$$\mathcal{I}(V_i, V_j) = \mathcal{I}(\gamma) , \quad (2.19)$$

where the function $\mathcal{I}(\gamma)$ for a path γ is defined by

$$\mathcal{I}(\gamma) \equiv \sum_{l=0}^{|\gamma|-1} \sigma^{(l)}(\gamma) \mathcal{I}(E^{(l)}(\gamma)) . \quad (2.20)$$

Here $\sigma^{(l)}(\gamma)$, $E^{(l)}(\gamma)$ and $|\gamma|$ denote respectively the l -th sign, l -th edge and the length of the path γ (see Appendix A for details).

2.6. Local analysis: asymptotic evaluation of elementary integral

The main theme in this section 2.6 is to evaluate asymptotically the elementary integral $\mathcal{I}(E_m) = \int_{E_m} e^{-f(s)/\epsilon} ds$ for each positive stationary phase path $E_m \in \mathbf{E}$.

We consider only a generic case in which no saddle connection occurs. In this case, every stationary phase path E_m has the unique saddle point $s_{\Pi, m}$ on it and hence the stationary phase condition (2.6) is rewritten as

$$\arg e^{-f(s)/\epsilon} = \arg e^{-f(s_{\Pi, m})/\epsilon} \quad \text{for any } s \in E_m . \quad (2.21)$$

Along a stationary phase path E_m , the magnitude of integrand $|e^{-f(s)/\epsilon}|$ takes the maximum just at the saddle point $s_{\Pi, m}$ on it. With use of this fact, the analytic

*) In this paper, the ‘‘path’’ is used in the two meanings; one is the path γ in the context of graph theory and the other is the path C along which some line integral is evaluated.

**) The notation $\Gamma(\mathbf{G}; V_i, V_j)$ denotes the set of paths in a graph \mathbf{G} that connect a vertex $V_i \in \mathbf{V}$ to a vertex $V_j \in \mathbf{V}$ (see Appendix A).

function $f(s)$ in the integrand $e^{-f(s)/\epsilon}$ is expanded around the saddle point $s_{\text{II},m}$ and the line integral $\mathcal{I}(E_m)$ along E_m is estimated by Gauss integral to have⁸⁾

$$\mathcal{I}(E_m) \sim \mathcal{I}_{\text{II}}(E_m) \equiv \sqrt{\frac{2\pi\epsilon}{|f''(s_{\text{II},m})|}} e^{i\theta(E_m)} e^{-f(s_{\text{II},m})/\epsilon}, \quad (2.22)$$

where $\theta(E_m)$ is the rotation angle of the positive stationary phase path E_m . According to our convention of positive direction for stationary phase paths, the rotation angle $\theta(E_m)$ must be in the range $-\frac{\pi}{2} < \theta(E_m) \leq \frac{\pi}{2}$ (2.10).

Furthermore, the rotation angle $\theta(E_m)$ for a stationary phase path E_m is expressed in terms of the saddle point $s_{\text{II},m}$ on it as^{*})

$$\theta(E_m) = -\frac{1}{2} \arg f''(s_{\text{II},m}) \quad \text{or} \quad \theta(E_m) = -\frac{1}{2} \arg f''(s_{\text{II},m}) + \pi \pmod{2\pi}. \quad (2.23)$$

When the branch of argument $\arg f''(s_{\text{II},m})$ in (2.23) is fixed to be

$$-\pi \leq \text{Arg } f''(s_{\text{II},m}) < \pi, \quad (2.24)$$

then the rotation angle $\theta(E_m)$ (2.23) is written uniquely as

$$\theta(E_m) = -\frac{1}{2} \text{Arg } f''(s_{\text{II},m}), \quad (2.25)$$

because of our convention $-\frac{\pi}{2} < \theta(E_m) \leq \frac{\pi}{2}$ (2.10).

We finally rewrite the asymptotic expression $\mathcal{I}_{\text{II}}(E_m)$ (2.22) of integral $\mathcal{I}(E_m)$ with use of the principal argument function $\text{Arg} \cdot$ (2.24) as^{**)}

$$\mathcal{I}_{\text{II}}(E_m) = \sqrt{\frac{2\pi\epsilon}{|f''(s_{\text{II},m})|}} e^{-i\frac{1}{2}\text{Arg } f''(s_{\text{II},m})} e^{-f(s_{\text{II},m})/\epsilon} \equiv \mathcal{G}(s_{\text{II},m}), \quad (2.26)$$

where $\mathcal{G}(s_{\text{II},m})$ is termed the *asymptotic contribution* from a saddle point $s_{\text{II},m}$.

2.7. Global analysis: asymptotic evaluation of a given integral

This section 2.7 combines the global analytic formula in §2.5 and the local analytic formula in §2.6 in order to derive the asymptotic expression $\mathcal{I}_{\text{II}}(V_i, V_j)$ of a given integral $\mathcal{I}(V_i, V_j) = \int_{V_i \rightarrow V_j} e^{-f(s)/\epsilon} ds$. This section 2.7 is given also as a summary of Section 2. In the following explanation, we assume that the saddle point equation $f'(s_{\text{II}}) = 0$ has an M solutions $\{s_{\text{II},m} : m = 0, 1, \dots, M-1\}$.

^{*}) When a point s on the path E_m is sufficiently close to the saddle point $s_{\text{II},m}$, the stationary phase condition $\arg e^{-f(s)/\epsilon} = \arg e^{-f(s_{\text{II},m})/\epsilon}$ (2.21) is reduced to $\Im \{f''(s_{\text{II},m})(s - s_{\text{II},m})^2\} = 0$, since ϵ is real. This equation immediately leads (2.23).

^{**)} In some textbooks, (2.26) is written in a shorter form as $\sqrt{\frac{2\pi\epsilon}{f''(s_{\text{II}})}} e^{-f(s_{\text{II}})/\epsilon}$ but it has an ambiguity that comes from the double-valuedness of square root. The specification of branch for this square root is necessary for us because the *switching line* is viewed as a branch cut arising from the double-valuedness of this square root.

2.7.1. Specification of positive stationary phase path

For every saddle point $s_{\text{II},m}$, the positive stationary phase path E_m is obtained by the two steps. First, if this stationary phase path E_m is considered just as a geometrical curve, this curve is specified by

$$\mathbf{E}_m \equiv \{s \in \mathbb{C} : \arg e^{-f(s)/\epsilon} = \arg e^{-f(s_{\text{II},m})/\epsilon}\}, \tag{2.27}$$

which is termed the *undirected* stationary phase path. Second, this undirected stationary phase path \mathbf{E}_m is given a direction to become the positive stationary phase path E_m so as to satisfy $-\frac{\pi}{2} < \theta(E_m) \leq \frac{\pi}{2}$.

2.7.2. Reduction to the connectivity graph

The configuration of positive stationary phase paths $\{E_m : m = 0, 1, \dots, M - 1\}$ on the complex s -plane $\{s \in \mathbb{C}\}$ is reduced to the connectivity graph $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \partial^-, \partial^+)$ according to our rule explained in §2.5.1.

2.7.3. Asymptotic evaluation of integral

The integral $\mathcal{I}(V_i, V_j)$ for valleys V_i and V_j is evaluated asymptotically along a path $\gamma \in \Gamma(\mathbf{G}; V_i, V_j)$ in the graph \mathbf{G} that connects the vertex $V_i \in \mathbf{V}$ to the vertex $V_j \in \mathbf{V}$ as

$$\mathcal{I}(V_i, V_j) \sim \mathcal{I}_{\text{II}}(V_i, V_j) = \mathcal{I}_{\text{II}}(\gamma) = \sum_{l=0}^{|\gamma|-1} \sigma^{(l)}(\gamma) \mathcal{I}_{\text{II}}(E^{(l)}(\gamma)), \tag{2.28}$$

where each local asymptotic evaluation $\mathcal{I}_{\text{II}}(E^{(l)}(\gamma))$ is given by (2.26).*

Any edge $E^{(l)}(\gamma)$ contained in a path $\gamma \in \Gamma(\mathbf{G}; V_i, V_j)$ is called the *contributing edge* to the path γ , whereas any edge that is *not* contained in the path γ is called the *non-contributing edge* to γ . Furthermore, any stationary phase path corresponding to a contributing (non-contributing) edge is called the *contributing (non-contributing) stationary phase path*. Similarly, any saddle point s_{II} that is on a contributing (non-contributing) stationary phase path is called the *contributing (non-contributing) saddle point*.

§3. Application to the Airy function

This section 3 derives the asymptotic expression of the Airy function $\text{Ai}(x)$. We utilize the procedure of steepest descent method, which is summarized at the previous section 2.7, in order to asymptotically evaluate the Airy function $\text{Ai}(x)$ for every value of $x \in \mathbb{C}$.

During this asymptotic evaluation, we will find that the connectivity graph $\mathbf{G}(x)$ as a function of x changes abruptly just when the control parameter x crosses the three types of boundary line (*saddle connection line*,⁶⁾ *branch cut*¹⁸⁾ and *switching line*). The former two lines are well-known as the basic ingredients in steepest

*) If the given integral contains an amplitude factor $a(s)$ as $\mathcal{I}(C) = \int_C a(s)e^{-f(s)/\epsilon} ds$, the local asymptotic evaluation becomes $\mathcal{I}_{\text{II}}(E) = a(s_{\text{II}}) \sqrt{\frac{2\pi\epsilon}{|f''(s_{\text{II}})|}} e^{-i\frac{1}{2}\text{Arg} f''(s_{\text{II}})} e^{-f(s_{\text{II}})/\epsilon}$.

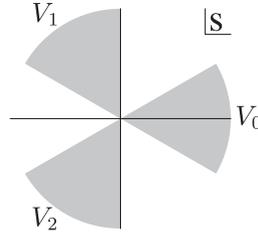


Fig. 4. Valleys $\{V_n : n = 0, 1, 2\}$ of the integrand $e^{-f(s;x)} = e^{-\frac{1}{3}s^3 + xs}$.

descent method. However, as far as we know, it is not written explicitly anywhere that the switching line, which is a new type of branch cut, is also necessary in steepest descent method to derive actually the asymptotic expression of a given integral. The appearance of switching line on the space of x can be viewed as one example of Berry’s adiabatic phase^{14),15)} (see §3.5).

3.1. Specification of valley

The Airy function $\text{Ai}(x)$ for a complex argument $x \in \mathbb{C}$ is defined by the integral representation¹⁶⁾

$$\text{Ai}(x) = \frac{1}{2\pi i} \int_C e^{-f(s;x)} ds \quad \text{with} \quad C : \infty e^{i4\pi/3} \rightarrow \infty e^{i2\pi/3}, \quad (3.1)$$

where the function $f(s;x)$ of the state variable $s \in \mathbb{C}$ and the control parameter $x \in \mathbb{C}$ is given by

$$f(s;x) \equiv \frac{1}{3}s^3 - xs. \quad (3.2)$$

Below, we let S and X denote respectively the entire space of state variable s and the entire space of control parameter x , i.e. $S \equiv \{s \in \mathbb{C}\}$ and $X \equiv \{x \in \mathbb{C}\}$.

For any value of control parameter $x \in X$, the integrand $e^{-f(s;x)}$ has three fixed valleys V_0, V_1 and V_2 , each of which is (see Fig. 4)

$$V_n = \left\{ s \in S : |s| \rightarrow \infty \wedge \left(n - \frac{1}{4}\right) \frac{2}{3}\pi < \arg s < \left(n + \frac{1}{4}\right) \frac{2}{3}\pi \right\}. \quad (3.3)$$

The integral from an initial valley V_i to a terminal valley V_j is denoted by $\mathcal{I}(V_i, V_j; x) = \int_{V_i \rightarrow V_j} e^{-f(s;x)} ds$. With use of this integral, the Airy function $\text{Ai}(x)$ (3.1) is represented as $\text{Ai}(x) = \frac{1}{2\pi i} \mathcal{I}(V_2, V_1; x)$.

3.2. Specification of saddle point

The saddle point $s_{\text{II}}(x)$ as a function of x is determined from

$$\text{(SE)} : \quad f'(s_{\text{II}}(x); x) = \{s_{\text{II}}(x)\}^2 - x = 0. \quad (3.4)$$

The analytic continuation of the local function $s_{\text{II}}(x)$ on the base space X yields a single-valued analytic function on a system of Riemann sheets \tilde{X} over X . We express this single-valued analytic function on \tilde{X} by

$$s_{\text{II}}(\tilde{x}) = -\sqrt{|\tilde{x}|} e^{i \arg(\tilde{x})/2} \quad \text{for} \quad \tilde{x} \in \tilde{X}, \quad (3.5)$$

where the control parameter in the covering space \tilde{X} is denoted by \tilde{x} to be distinguished from the control parameter x in the base space X . The system of Riemann sheets \tilde{X} consists of two Riemann sheets

$$X_0 = \{\tilde{x} : -\pi \leq \arg \tilde{x} < \pi\} \quad \text{and} \quad X_1 = \{\tilde{x} : \pi \leq \arg \tilde{x} < 3\pi\}, \quad (3.6)$$

which are copies of the base space X . The 0th Riemann sheet X_0 has one branch cut $\tilde{B}_0 = \{\tilde{x} \in X_0 : \arg \tilde{x} = -\pi\}$ on it, whereas the 1st Riemann sheet X_1 has one branch cut $\tilde{B}_1 = \{\tilde{x} \in X_1 : \arg \tilde{x} = \pi\}$ on it.

3.3. Specification of undirected stationary phase path

For a value of control parameter \tilde{x} in the Riemann surface \tilde{X} , the undirected stationary phase path $\mathbf{E}(\tilde{x})$ passing through the saddle point $s_{\text{II}}(\tilde{x})$ is specified by (see (2.27))

$$\mathbf{E}(\tilde{x}) = \{s \in \mathbb{C} : \arg e^{-f(s;\tilde{x})} = \arg e^{-f(s_{\text{II}}(\tilde{x});\tilde{x})}\}. \quad (3.7)$$

Figure 5 depicts the undirected stationary phase path $\mathbf{E}(\tilde{x})$ on the state variable space S for some typical values of control parameter \tilde{x} in the Riemann surface \tilde{X} .

Next, we trace the change in two end valleys $\partial(\tilde{x})\mathbf{E}(\tilde{x})$ of undirected stationary phase path $\mathbf{E}(\tilde{x})$ while the control parameter \tilde{x} is varied on the Riemann surface \tilde{X} . The end valleys of it are read from Fig. 5 to be

$$\partial(\tilde{x})\mathbf{E}(\tilde{x}) = \begin{cases} \{V_1, V_2\} & \tilde{x} \in \tilde{D}_0, \\ \{V_0, V_2\} & \tilde{x} \in \tilde{D}_1, \\ \{V_0, V_1\} & \tilde{x} \in \tilde{D}_2, \end{cases} \quad (3.8)$$

where the domains \tilde{D}_0, \tilde{D}_1 and \tilde{D}_2 in the Riemann surface \tilde{X} are defined by

$$\begin{aligned} \tilde{D}_0 &= \{\tilde{x} \in \tilde{X} : -\frac{2\pi}{3} < \arg \tilde{x} < \frac{2\pi}{3} \pmod{4\pi}\}, \\ \tilde{D}_1 &= \{\tilde{x} \in \tilde{X} : \frac{2\pi}{3} < \arg \tilde{x} < 2\pi \pmod{4\pi}\}, \\ \tilde{D}_2 &= \{\tilde{x} \in \tilde{X} : 2\pi < \arg \tilde{x} < \frac{10\pi}{3} \pmod{4\pi}\}. \end{aligned} \quad (3.9)$$

The equation (3.8) means that the end valleys of undirected stationary phase path $\mathbf{E}(\tilde{x})$ changes abruptly just when the control parameter \tilde{x} crosses every boundary line between two adjacent domains in (3.9). These boundary lines, which are called the *saddle connection lines*, are expressed as

$$\begin{aligned} \tilde{L}_0 &= \{\tilde{x} \in \tilde{X} : \arg \tilde{x} = 2\pi \pmod{4\pi}\}, \\ \tilde{L}_1 &= \{\tilde{x} \in \tilde{X} : \arg \tilde{x} = \frac{2\pi}{3} \pmod{4\pi}\}, \\ \tilde{L}_2 &= \{\tilde{x} \in \tilde{X} : \arg \tilde{x} = -\frac{2\pi}{3} \pmod{4\pi}\}. \end{aligned} \quad (3.10)$$

3.4. Specification of positive stationary phase path

In this section 3.4, the undirected stationary phase path $\mathbf{E}(\tilde{x})$ is given a direction to become the positive stationary phase path $E(\tilde{x})$. According to our convention (2.10) of the positive direction, the direction of $E(\tilde{x})$ is specified such that its rotation

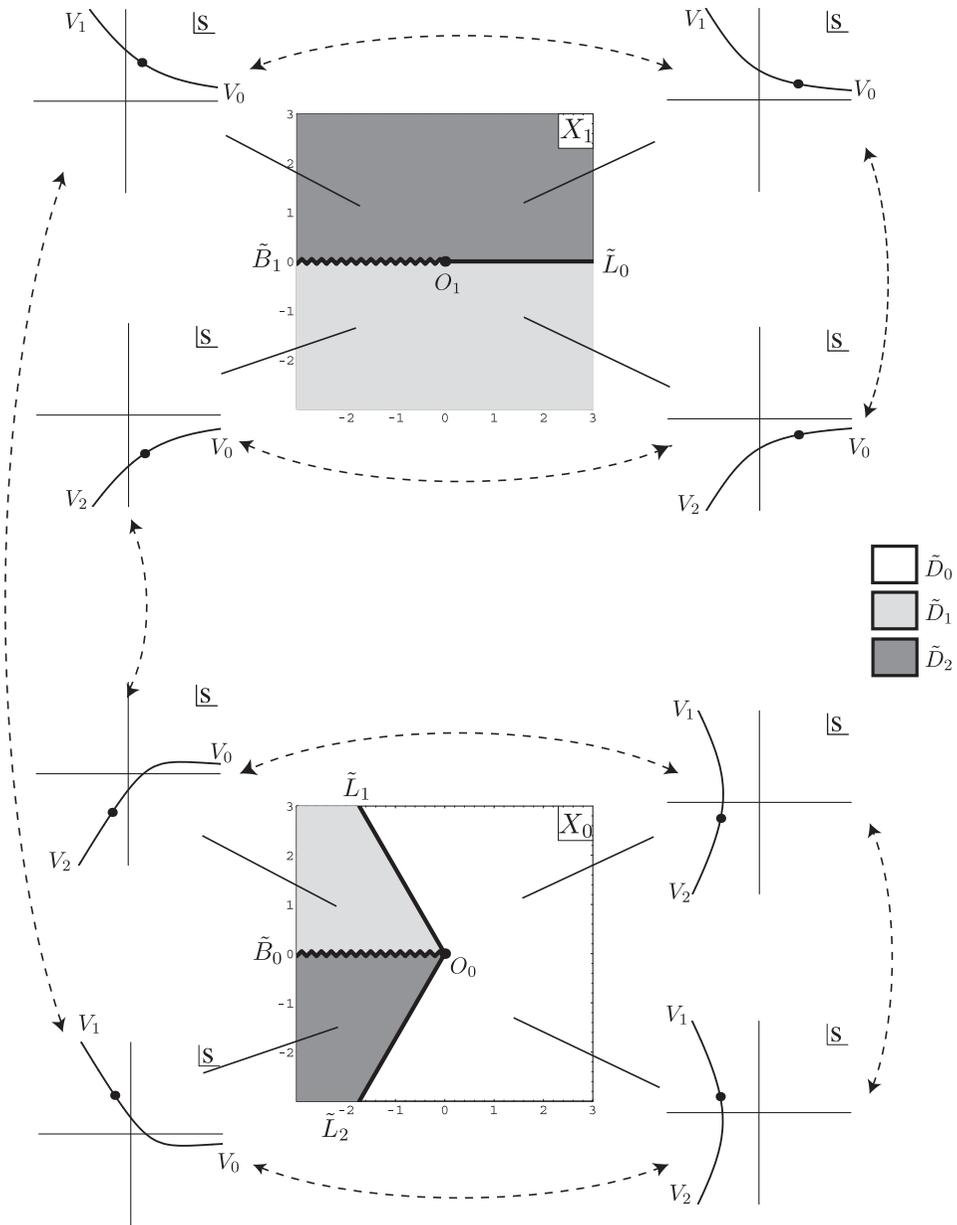


Fig. 5. Undirected stationary phase path $E(\tilde{x})$ (solid curve) for some typical value of $\tilde{x} \in \tilde{X}$. The domain in \tilde{X} that is white, light gray or dark gray is named respectively \tilde{D}_0 , \tilde{D}_1 or \tilde{D}_2 . The dot (\bullet) in the state variable space S indicates the saddle point $s_{II}(\tilde{x})$.

angle $\theta(E(\tilde{x}))^*$ is in the range

$$-\frac{\pi}{2} < \theta(E(\tilde{x})) \leq \frac{\pi}{2} . \quad (3.11)$$

*) The rotation angle $\theta(E)$ of a stationary phase path E denotes the geometrical angle over the state variable space S from the positive $\Re s$ direction to the tangential direction of E at the saddle point s_{II} in anticlockwise rotation (see also Fig. 2).

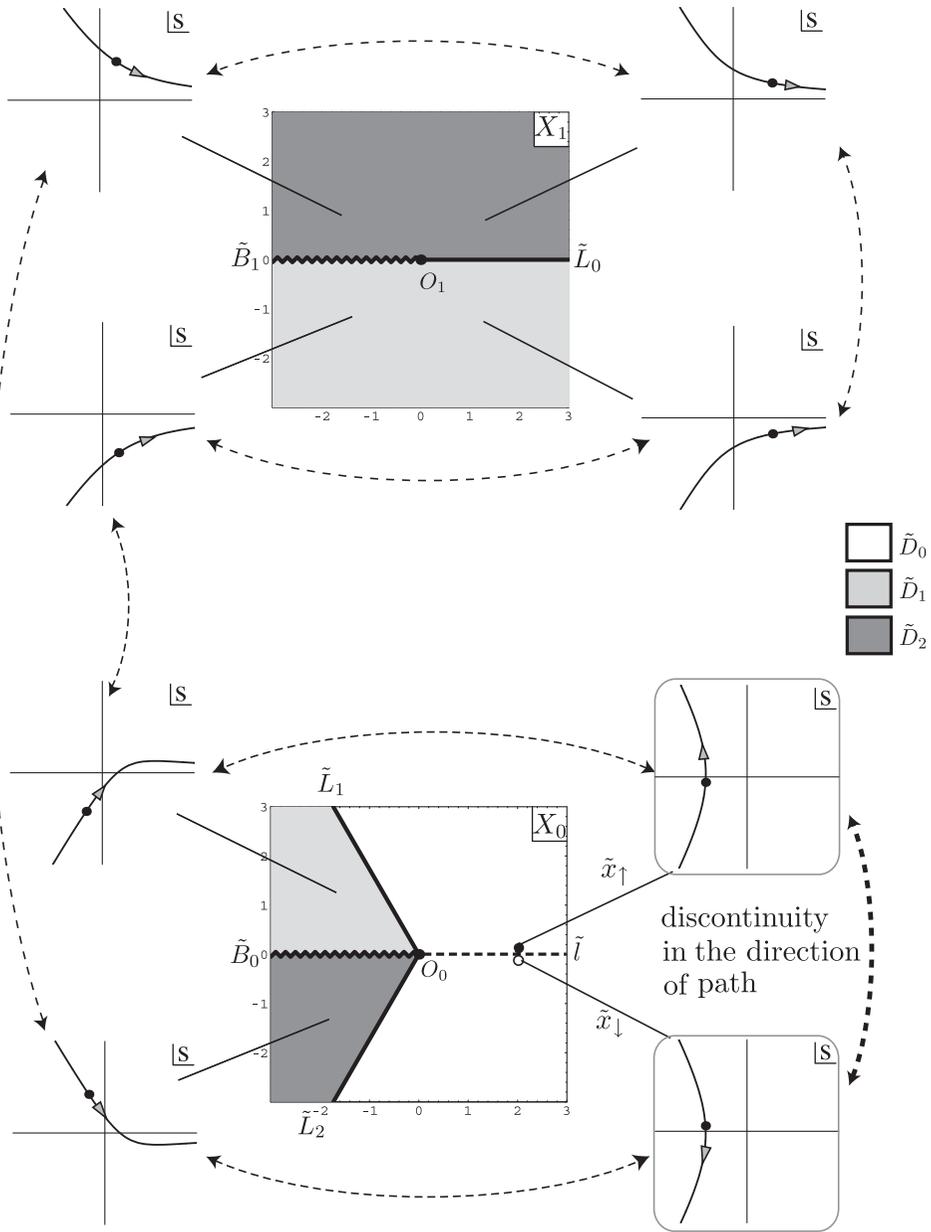


Fig. 6. Positive stationary phase path $E(\tilde{x})$ (the arrow in S) for some typical value of $\tilde{x} \in \tilde{X}$.

With use of this convention, the undirected stationary phase path $E(\tilde{x})$ depicted in Fig. 5 is given a certain direction as shown in Fig. 6.

The main purpose of Fig. 6 is to show that the rotation angle $\theta(E(\tilde{x}))$ of the positive stationary phase path $E(\tilde{x})$ exhibits a discontinuity when the control parameter \tilde{x} crosses the half-line

$$\tilde{l} = \{ \tilde{x} \in X_0 : \arg \tilde{x} = 0 \} , \tag{3.12}$$

which is named the *switching line*. Mathematically this fact is written as

$$\theta(E(\tilde{x}_\downarrow)) = \theta(E(\tilde{x}_\uparrow)) - \pi \pmod{2\pi}, \quad (3.13)$$

where the point $\tilde{x}_\uparrow = R e^{i\delta}$ ($R > 0$ and $\delta \gtrsim 0$) is slightly above the switching line $\tilde{l} \in X_0$ and the point $\tilde{x}_\downarrow = R e^{-i\delta}$ is slightly below it.

The reader may consider that this abrupt change in the direction $\theta(E(\tilde{x}))$ of $E(\tilde{x})$ is mere artifact arising from our specific convention of positive direction (3.11). However, the double-valuedness of the direction $\theta(E(\tilde{x}))$ is intrinsic and unavoidable because, no matter how we define the positive direction, the direction $\theta(E(\tilde{x}))$ of $E(\tilde{x})$ becomes opposite after the control parameter \tilde{x} is moved around once the origin O_0 of 0th Riemann sheet X_0 and once the origin O_1 of 1st Riemann sheet X_1 . In fact, this phenomenon can be viewed as one example of Berry's adiabatic phase^{14),15)} (see the next section 3.5).

The equation (3.13) means that *four* (not two) Riemann sheets X_0, X_1 and X'_0, X'_1 are required to express the rotation angle $\theta(E(\tilde{x}))$ of the positive stationary phase path $E(\tilde{x})$ in a single-valued continuous function on them. The switching line \tilde{l} (3.12) is the branch cut that sews $\tilde{X} \equiv X_0 \cup X_1$ and $\tilde{X}' \equiv X'_0 \cup X'_1$ together to obtain the domain $X_0 \cup X_1 \cup X'_0 \cup X'_1$ of a continuous single-valued function $\theta(E(\tilde{x}))$.

3.5. Interpretation of the appearance of switching line as Berry's adiabatic phase

We here note the relation between the switching line \tilde{l} and *Berry's adiabatic phase*.^{14),15)} On the one hand, the switching line \tilde{l} is the branch cut on the Riemann surface \tilde{X} to make the rotation angle $\theta(E(\tilde{x}))$ of the positive stationary phase path $E(\tilde{x})$ be a single-valued function of control parameter $\tilde{x} \in \tilde{X}$. On the other hand, Berry's adiabatic phase^{14),15)} arises in a variety of wave systems whenever one can externally control at least two parameters affecting a wave; the occurrence of Berry's adiabatic phase in a wave system indicates that the parameter dependence of the system is singular at some point in the control parameter space.

This section 3.5 explains that the above discontinuity in the rotation angle $\theta(E(\tilde{x}))$ over the Riemann surface \tilde{X} is interpreted as an occurrence of Berry's adiabatic phase.

3.5.1. Occurrence of Berry's adiabatic phase in the application of steepest descent method to a general integral

We first show that Berry's adiabatic phase arises in the application of steepest descent method to a general integral of the form

$$\mathcal{I}(\tilde{x}) = \int_C e^{-f(s;\tilde{x})} ds, \quad (3.14)$$

where the function $f(s;\tilde{x})$ is at least cubic with respect to the state variable space s .*) Here, \tilde{x} is the control parameter to express the saddle point $s_{\text{II}}(\tilde{x})$ of the integrand $e^{-f(s;\tilde{x})}$ in a single-valued manner on the whole space \tilde{X} of the parameter \tilde{x} .

*) This assumption is necessary in order that the 2nd-order derivative $f''(s;\tilde{x})$ of $f(s;\tilde{x})$ with respect to the state variable s is not a constant with s .

We focus our attention to the vicinity of saddle point $s_{\text{II}}(\tilde{x})$. There, the function $f(s; \tilde{x})$ is approximated to be

$$f(s; \tilde{x}) \approx f(s_{\text{II}}(\tilde{x}); \tilde{x}) + Q \quad \text{with} \quad Q = \frac{1}{2} f''(s_{\text{II}}(\tilde{x}); \tilde{x}) (s - s_{\text{II}}(\tilde{x}))^2 . \quad (3.15)$$

We rewrite the coefficient $\frac{1}{2} f''(s_{\text{II}}(\tilde{x}); \tilde{x})$ of the quadratic term Q in terms of two real quantities $A(\tilde{x}) > 0$ and $\phi(\tilde{x}) \in [-\pi, \pi)$ as

$$\frac{1}{2} f''(s_{\text{II}}(\tilde{x}); \tilde{x}) = A(\tilde{x}) e^{i\phi(\tilde{x})} . \quad (3.16)$$

In addition, the difference between the point $s \in E(\tilde{x})$ and the saddle point $s_{\text{II}}(\tilde{x})$ is denoted by $\xi(s; \tilde{x}) + i\eta(s; \tilde{x})$, i.e.

$$s - s_{\text{II}}(\tilde{x}) = \xi(s; \tilde{x}) + i\eta(s; \tilde{x}) , \quad (3.17)$$

where both of $\xi(s; \tilde{x})$ and $\eta(s; \tilde{x})$ are real. Then, the quadratic term Q (3.15) is rewritten as

$$Q = A(\tilde{x}) \{(\mathbf{v}, M_{\text{r}}\mathbf{v}) + i(\mathbf{v}, M_{\text{i}}\mathbf{v})\} , \quad (3.18)$$

where

$$M_{\text{r}} \equiv \begin{pmatrix} \cos \phi(\tilde{x}) & -\sin \phi(\tilde{x}) \\ -\sin \phi(\tilde{x}) & -\cos \phi(\tilde{x}) \end{pmatrix}, \quad M_{\text{i}} \equiv \begin{pmatrix} \sin \phi(\tilde{x}) & \cos \phi(\tilde{x}) \\ \cos \phi(\tilde{x}) & -\sin \phi(\tilde{x}) \end{pmatrix}, \quad \mathbf{v} \equiv \begin{pmatrix} \xi(s; \tilde{x}) \\ \eta(s; \tilde{x}) \end{pmatrix} . \quad (3.19)$$

We now express the vector \mathbf{v} on the state variable space S as a linear combination of two eigenvectors of the matrix M_{r} (3.19), i.e.

$$\mathbf{v} = c_{+}\mathbf{e}_{+}(\tilde{x}) + c_{-}\mathbf{e}_{-}(\tilde{x}) , \quad (3.20)$$

where the eigenvectors

$$\mathbf{e}_{+}(\tilde{x}) = \begin{pmatrix} \cos \frac{\phi(\tilde{x})}{2} \\ -\sin \frac{\phi(\tilde{x})}{2} \end{pmatrix} \quad \text{and} \quad \mathbf{e}_{-}(\tilde{x}) = \begin{pmatrix} \sin \frac{\phi(\tilde{x})}{2} \\ \cos \frac{\phi(\tilde{x})}{2} \end{pmatrix} \quad (3.21)$$

satisfy the equations $M_{\text{r}}\mathbf{e}_{+} = \mathbf{e}_{+}$, $M_{\text{r}}\mathbf{e}_{-} = -\mathbf{e}_{-}$, $M_{\text{i}}\mathbf{e}_{+} = \mathbf{e}_{-}$, $M_{\text{i}}\mathbf{e}_{-} = \mathbf{e}_{+}$ and $(\mathbf{e}_{+}, \mathbf{e}_{-}) = 0$. By using these equations, the quadratic term Q (3.18) is further rewritten as

$$Q = A(\tilde{x})(c_{+} + ic_{-})^2 . \quad (3.22)$$

Here, it can be shown that^{*)} the direction $-\frac{\phi(\tilde{x})}{2}$ of the former eigenvector $\mathbf{e}_{+}(\tilde{x})$ over the state variable space S is equal to one of two steepest descent directions of the magnitude of integrand $|e^{-f(s; \tilde{x})}|$ that go away from $s_{\text{II}}(\tilde{x})$. Furthermore, the direction $\theta(E(\tilde{x}))$ of the positive stationary phase path $E(\tilde{x})$ is also equal to one of two steepest descent directions of $|e^{-f(s; \tilde{x})}|$ going away from $s_{\text{II}}(\tilde{x})$ (see §2.2.2).

^{*)} When the vector \mathbf{v} traces the direction of the former eigenvector $\mathbf{e}_{+}(\tilde{x})$ as $\mathbf{v} = c_{+}\mathbf{e}_{+}(\tilde{x})$, the integrand $e^{-f(s; \tilde{x})}$ is approximated to be $e^{-f(s; \tilde{x})} \approx e^{-f(s_{\text{II}}(\tilde{x}); \tilde{x})} e^{-A(\tilde{x})c_{+}^2}$ (because of (3.15), (3.22) and $c_{-} = 0$) and it attenuates in the steepest descent manner.

These sentences mean that the direction $-\frac{\phi(\tilde{x})}{2}$ of the eigenvector $\mathbf{e}_+(\tilde{x})$ is parallel or anti-parallel to the rotation angle $\theta(E(\tilde{x}))$ of $E(\tilde{x})$, i.e.

$$\theta(E(\tilde{x})) = -\frac{\phi(\tilde{x})}{2} \quad \text{or} \quad \theta(E(\tilde{x})) = -\frac{\phi(\tilde{x})}{2} + \pi \quad \text{with} \quad \phi \in [-\pi, \pi) . \quad (3.23)$$

According to our convention $-\frac{\pi}{2} < \theta(E(\tilde{x})) \leq \frac{\pi}{2}$ (3.11) of the positive direction for stationary phase path, the former case is selected as

$$\theta(E(\tilde{x})) = -\frac{\phi(\tilde{x})}{2} . \quad (3.24)$$

This equation (3.24) yields the following conclusion. When the control parameter \tilde{x} moves around once a point \tilde{x}_* where the 2nd-order differentiation $f''(s_{\text{II}}(\tilde{x}); \tilde{x})$ of the function $f(s; \tilde{x})$ vanishes, then the argument $\phi(\tilde{x})$ of $f''(s_{\text{II}}(\tilde{x}); \tilde{x})$ changes by 2π . In this variation of control parameter \tilde{x} , the rotation angle $\theta(E(\tilde{x})) = -\frac{\phi(\tilde{x})}{2}$ (3.24) changes just by π (not 2π). This discontinuity in the rotation angle $\theta(E(\tilde{x}))$ clearly comes from the double-valuedness of the eigenvector $\mathbf{e}_+(\tilde{x})$ (3.21) of the matrix $M_{\tilde{x}}$ (3.19) as a function of $\phi(\tilde{x})$; this double-valuedness is well-known^{14),15)} as the *real symmetric case* of Berry's adiabatic phase. Thus, the discontinuity in the rotation angle $\theta(E(\tilde{x}))$ of the positive stationary phase path $E(\tilde{x})$ can be interpreted as an occurrence of Berry's adiabatic phase.

3.5.2. Occurrence of Berry's adiabatic phase in the application of steepest descent method to the Airy function

We now return to the asymptotic analysis of the Airy function $\text{Ai}(x)$, in which the phase function $f(s; x)$ is written as $f(s; x) = \frac{1}{3}s^3 - xs$. In this analysis, the equation $\frac{1}{2}f''(s_{\text{II}}(\tilde{x}); \tilde{x}) = A(\tilde{x}) e^{i\phi(\tilde{x})}$ (3.16) simply becomes $s_{\text{II}}(\tilde{x}) = A(\tilde{x}) e^{i\phi(\tilde{x})}$ and hence the quantity $\phi(\tilde{x})$ represents the phase of the saddle point $s_{\text{II}}(\tilde{x})$. According to the mathematical expression $s_{\text{II}}(\tilde{x}) = -\sqrt{|\tilde{x}|} e^{i \arg(\tilde{x})/2}$ (3.5) of the saddle point $s_{\text{II}}(\tilde{x})$, its phase $\phi(\tilde{x})$ is written as

$$\phi(\tilde{x}) = \frac{1}{2} \arg \tilde{x} - \pi \quad \text{mod } 2\pi \quad \text{for} \quad \tilde{x} \in \tilde{X} . \quad (3.25)$$

Then, the rotation angle $\theta(E(\tilde{x})) = -\frac{\phi(\tilde{x})}{2}$ (3.24) of the positive stationary phase path $E(\tilde{x})$ is written as

$$\theta(E(\tilde{x})) = -\frac{1}{4} \arg \tilde{x} + \frac{\pi}{2} . \quad (3.26)$$

When the argument $\arg \tilde{x}$ of control parameter \tilde{x} is varied by 4π over the Riemann surface $\tilde{X} = X_0 \cup X_1$, the phase $\phi(\tilde{x})$ (3.25) of the saddle point $s_{\text{II}}(\tilde{x})$ changes by 2π but the rotation angle $\theta(E(\tilde{x})) = -\frac{\phi(\tilde{x})}{2}$ (3.26) of the path $E(\tilde{x})$ changes just by π . This discontinuity in the rotation angle $\theta(E(\tilde{x}))$ over the Riemann surface \tilde{X} is already demonstrated in Fig. 6. The switching line \tilde{l} is necessary on the Riemann surface \tilde{X} as one type of branch cut to make the rotation angle $\theta(E(\tilde{x}))$ of $E(\tilde{x})$ be a single-valued function of the control parameter $\tilde{x} \in \tilde{X}$.

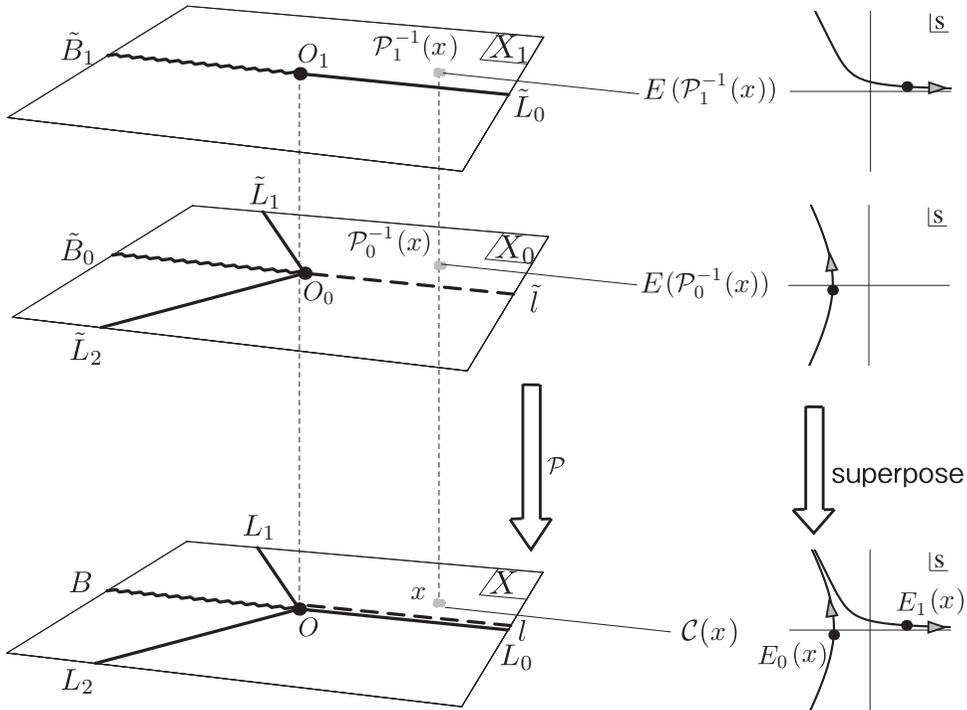


Fig. 7. Projection of two Riemann sheets X_0 and X_1 onto the base space X . The configuration $\mathcal{C}(x)$ of positive stationary phase paths for a value of control parameter $x \in X$ is obtained by superposing two positive stationary phase paths $E_0(x) \equiv E(\mathcal{P}_0^{-1}(x))$ and $E_1(x) \equiv E(\mathcal{P}_1^{-1}(x))$, which are attached to the inverse projections $\mathcal{P}_0^{-1}(x)$ and $\mathcal{P}_1^{-1}(x)$ of the point x .

3.6. Projection of two Riemann sheets onto the base space

3.6.1. Projection map

Up to now, we have considered the Riemann surface $\tilde{X} = X_0 \cup X_1$ on which the saddle point $s_{II}(\tilde{x})$ (3-5) is expressed as a single-valued function. On the other hand, the Airy function $\text{Ai}(x)$ itself, which is our target integral for asymptotic analysis, is defined over the base space $X = \{x \in \mathbb{C}\}$ and thereby is single-valued there. In order to relate the Riemann surface \tilde{X} to the base space $X = \{x \in \mathbb{C}\}$, we introduce the projection of the system of Riemann sheets \tilde{X} onto the base space X .

We let $\mathcal{P} : \tilde{X} \rightarrow X$ denote the projection map from the Riemann surface \tilde{X} to the base space X . When this projection \mathcal{P} is restricted to either Riemann sheet X_m ($m = 0, 1$), the restriction $\mathcal{P}|_{X_m} : X_m \rightarrow X$ becomes a one-to-one map. The inverse of this one-to-one map $\mathcal{P}|_{X_m}$ becomes the inverse projection from the base space $X = \{x \in \mathbb{C}\}$ to the m -th Riemann sheet X_m . This inverse map is denoted by $\mathcal{P}_m^{-1} \equiv (\mathcal{P}|_{X_m})^{-1} : X \rightarrow X_m$.

3.6.2. Superposition of positive stationary phase paths

In this section 3.6.2, we define the positive stationary phase path $E(x)$ as a function of the control parameter x on the base space X (not on the Riemann surface \tilde{X}). A given value of x on the base space X is lifted up to the two inverse projections

$\mathcal{P}_0^{-1}(x)$ and $\mathcal{P}_1^{-1}(x)$ on the two Riemann sheets X_0 and X_1 . Each inverse projection $\mathcal{P}_m^{-1}(x)$ with $m = 0, 1$ specifies the positive stationary phase path $E(\mathcal{P}_m^{-1}(x))$ in the state variable space S at that value $\mathcal{P}_m^{-1}(x)$ of control parameter \tilde{x} in the Riemann surface \tilde{X} . Each positive stationary phase path $E(\mathcal{P}_m^{-1}(x))$ is simply denoted by $E_m(x)$, i.e.

$$E_m(x) \equiv E(\mathcal{P}_m^{-1}(x)) . \quad (3.27)$$

Furthermore, we superpose the two positive stationary phase paths $E_0(x)$ and $E_1(x)$ as geometrical figures in the state variable space S . The superposed paths are symbolically denoted by

$$\mathcal{C}(x) \equiv (E_0(x), E_1(x)) , \quad (3.28)$$

which is called the *configuration* of positive stationary phase paths.

Subsequently, the saddle point $s_{\text{II}}(\mathcal{P}_m^{-1}(x))$ on an m -th positive stationary phase path $E_m(x) \equiv E(\mathcal{P}_m^{-1}(x))$ is also denoted by $s_{\text{II},m}(x)$. From the expression of the saddle point $s_{\text{II}}(\tilde{x})$ (3.5) for a point \tilde{x} on the Riemann surface \tilde{X} , the m -th saddle point $s_{\text{II},m}(x) \equiv s_{\text{II}}(\mathcal{P}_m^{-1}(x))$ is written as

$$s_{\text{II},m}(x) = (-1)^{m+1} |x|^{1/2} e^{i \text{Arg}(x)/2} \quad \text{with } -\pi \leq \text{Arg}(x) < \pi . \quad (3.29)$$

3.6.3. Three types of boundary line

The configuration $\mathcal{C}(x)$ of positive stationary phase paths can change discontinuously only when the control parameter $x \in X$ crosses the three types of boundary line. These boundary lines are the projections of branch cut, switching line and saddle connection line. In the application to the Airy function $\text{Ai}(x)$, there are 5 boundary lines in total as explained below.

- The branch cuts \tilde{B}_0 and \tilde{B}_1 in Riemann surface \tilde{X} are projected onto the identical line $B \equiv \mathcal{P}(\tilde{B}_0) = \mathcal{P}(\tilde{B}_1) = \{x \in X : \arg x = -\pi\}$, which is called the *branch cut in base space*.
- The switching line \tilde{l} in Riemann surface \tilde{X} is projected onto the line $l \equiv \mathcal{P}(\tilde{l}) = \{x \in X : \arg x = 0\}$, which is called the *switching line in base space*.
- Each saddle connection line \tilde{L}_n ($n = 0, 1, 2$) in Riemann surface \tilde{X} is projected onto the line $L_n \equiv \mathcal{P}(\tilde{L}_n) = \{x \in X : \arg x = \frac{2\pi}{3}n\}$, which is called the *saddle connection line in base space*.

3.6.4. Atomic domains and configurations on them

We have introduced three types of boundary line on base space X , the branch cut B , the switching line l and the saddle connection line L_n ($n = 0, 1, 2$). By these boundary lines, the base space X is divided into multiple domains. Each resultant domain obtained from this division is termed the *atomic domain* and is denoted by D . The configuration $\mathcal{C}(x)$ of positive stationary phase paths never changes discontinuously when x is in a single atomic domain D .

In the present application to the Airy function $\text{Ai}(x)$, there are four atomic domains D_0, D_1, D_2, D_3 .^{*)} For every atomic domain, we pick up one representative

^{*)} Generally speaking, *five* atomic domains appear around a fold degenerating point, at which some two saddle points collide. In the present application, however, one saddle connection line L_0

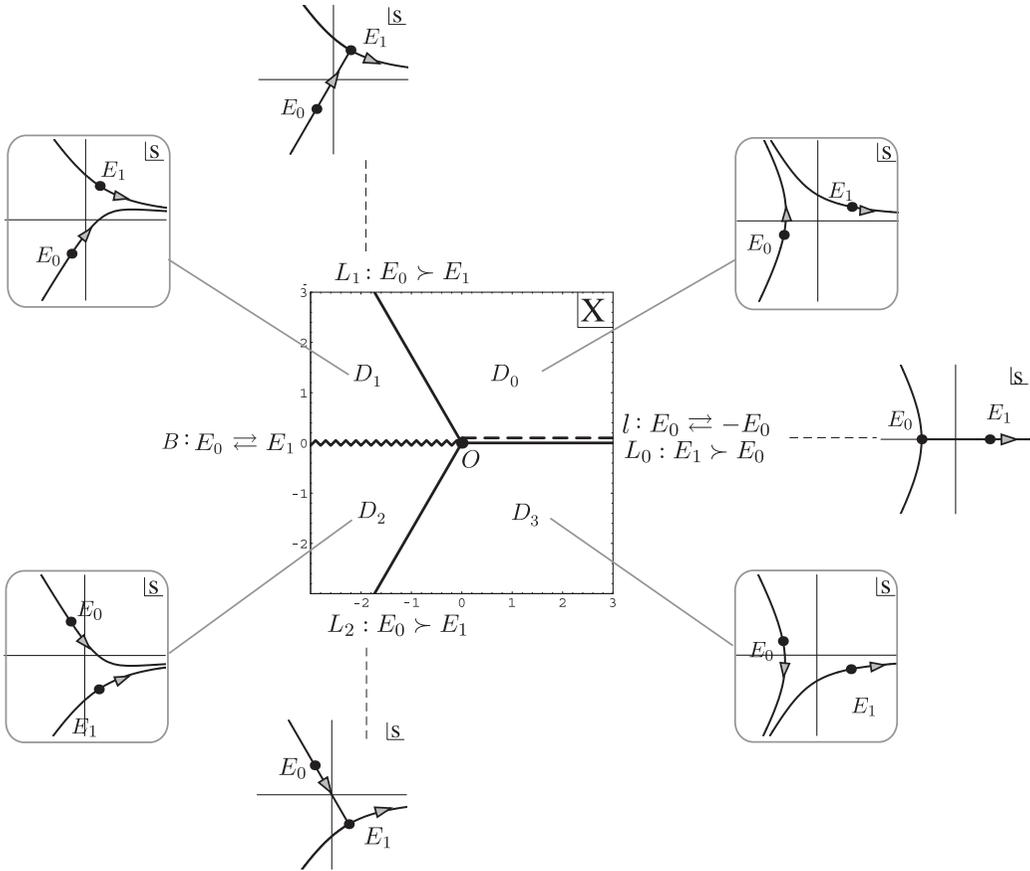


Fig. 8. Configuration $\mathcal{C}(x)$ of positive stationary phase paths for each atomic domain D_a ($a = 0, 1, 2, 3$). The configuration $\mathcal{C}(x)$ just when x is on each saddle connection line L_n ($n = 0, 1, 2$) is also depicted. When crossing the switching line l , the direction of the path $E_0(x)$ is suddenly reversed.

point x in it to depict the configuration $\mathcal{C}(x)$ of positive stationary phase paths on the state variable space S (see Fig. 8). In that figure, we observe various abrupt changes in the configuration $\mathcal{C}(x)$ when the control parameter $x \in X$ crosses every boundary line of two adjacent atomic domains.

First, when crossing the branch cut B , the *entities* of saddle points that are pointed to by the *names* $s_{II,0}(x)$ and $s_{II,1}(x)$ are suddenly exchanged. Corresponding to this, the *entities* of positive stationary phase paths that are pointed to by the *names* $E_0(x)$ and $E_1(x)$ are also exchanged when crossing the branch cut B . These properties when crossing the branch cut B are symbolically displayed as

$$B : s_{II,0} \rightleftharpoons s_{II,1} \quad \text{and} \quad B : E_0 \rightleftharpoons E_1 . \tag{3.30}$$

Second, when the control parameter x crosses the switching line l , the direction $\theta(E_0(x))$ of the 0th positive stationary phase path $E_0(x)$ is suddenly reversed. This is

and the switching line l accidentally coincide and thereby four (not five) atomic domains appear around the degenerating point $x = 0$.

caused by the discontinuity (3.13) in the direction $\theta(E(\tilde{x}))$ of the positive stationary phase path $E(\tilde{x})$ when the control parameter \tilde{x} in Riemann surface \tilde{X} crosses the original switching line \tilde{l} (3.12).

We can reinterpret this sudden reverse of the direction $\theta(E_0(x))$ of the 0th positive stationary phase path $E_0(x)$ when crossing the switching line l as follows. When the control parameter x crosses the switching line l , the names $E_0(x)$ and $-E_0(x)$ of two stationary phase paths in opposite directions exchange the entities of stationary phase paths that are pointed to by these names. This event is symbolically displayed as

$$l : E_0 \rightleftharpoons -E_0 . \quad (3.31)$$

We finally explain an abrupt change of the configuration $\mathcal{C}(x)$ when crossing a saddle connection line L_n , which is well-known as the saddle connection phenomenon.⁶⁾ For explanation, we focus our attention to the configuration $\mathcal{C}(x)$ when x is around the 1st saddle connection line L_1 .

Just when crossing the saddle connection line L_1 , the stationary phase path $E_0(x)$ passing through one saddle point $s_{\text{II},0}(x)$ hits the other saddle point $s_{\text{II},1}(x)$. This means that one of two steepest descent half-lines of the magnitude of integrand $|e^{-f(s;x)}|$ that go away from $s_{\text{II},0}(x)$ hits the other saddle point $s_{\text{II},1}(x)$. Hence in the neighborhood of saddle connection line L_1 , the magnitude of integrand $|e^{-f(s_{\text{II},1}(x);x)}|$ evaluated at the 1st saddle point $s_{\text{II},1}(x)$ is exponentially smaller than the magnitude of integrand $|e^{-f(s_{\text{II},0}(x);x)}|$ evaluated at the 0th saddle point $s_{\text{II},0}(x)$. In this meaning, the 0th saddle point $s_{\text{II},0}(x)$ is called the *dominant saddle point* on L_1 , whereas the 1st saddle point $s_{\text{II},1}(x)$ is called the *sub-dominant saddle point* on it.

From the correspondence between the name of saddle point $s_{\text{II},m}(x)$ and the name of stationary phase path $E_m(x)$, the 0th stationary phase path $E_0(x)$ is called the *dominant stationary phase path (dominant edge)* on the saddle connection line L_1 , whereas the 1st stationary phase path $E_1(x)$ is called the *sub-dominant stationary phase path (sub-dominant edge)* on it. This relation of dominance is symbolically expressed as

$$L_1 : E_0 \succ E_1 . \quad (3.32)$$

Similarly, the relations of dominance on the other saddle connection lines L_2 and L_0 hold as $L_2 : E_0 \succ E_1$ and $L_0 : E_1 \succ E_0$.

3.7. Reduction to the connectivity graph

The configuration $\mathcal{C}(x)$ for a representative point x in every atomic domain D is now reduced to the connectivity graph $G(x)$. The set of vertices $\mathbf{V} = \{V_0, V_1, V_2\}$ and the set of edges $\mathbf{E} = \{E_0, E_1\}$ in the graph G are constants independent of x . On the other hand, the incidence relations $\partial^\pm(x) : \mathbf{V} \rightarrow \mathbf{E}$ do depend on x but they take some constant values over each atomic domain. Hence, the connectivity graph $G(x) = (\mathbf{V}, \mathbf{E}, \partial^-(x), \partial^+(x))$ also takes a constant value over each atomic domain. The constant value $G(x)$ over every atomic domain is depicted as a graph in Fig. 9.

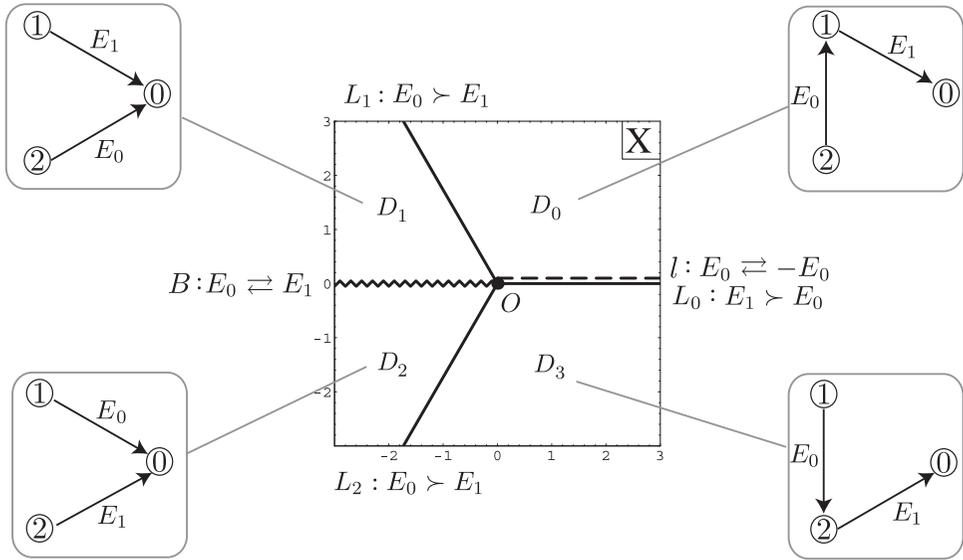


Fig. 9. Connectivity graph $G(x)$ for every atomic domain. When crossing the branch cut B , the names of two edges $E_0, E_1 \in E$ are abruptly exchanged. When crossing the switching line l , the direction of the 0th edge $E_0 \in E$ is suddenly reversed. When crossing a saddle connection line $L_n : E^{\text{dominant}} \succ E^{\text{sub-dominant}}$, an end vertex of the dominant edge E^{dominant} abruptly changes.

3.8. Asymptotic evaluation of integral

3.8.1. Asymptotic expression of the Airy function

According to the procedure of steepest descent method described in §2.7, the Airy function $\text{Ai}(x) = \frac{1}{2\pi i} \mathcal{I}(V_2, V_1; x)$ is asymptotically evaluated along a path $\gamma(x) \in \Gamma(G(x); V_2, V_1)$ in the connectivity graph $G(x)$ that connects the initial vertex $V_2 \in V$ to the terminal vertex $V_1 \in V$. The *simple path*^{*)} $\gamma(x)$ from V_2 to V_1 when x is in each atomic domain is easily read from Fig. 9 as

$$\gamma(x) = \begin{cases} (V_2, +1, E_0, V_1) & x \in D_0 \\ & \text{--- } L_1 : E_0 \succ E_1 , \\ (V_2, +1, E_0, V_0, -1, E_1, V_1) & x \in D_1 \\ & \text{--- } B : E_0 \rightleftharpoons E_1 , \\ (V_2, +1, E_1, V_0, -1, E_0, V_1) & x \in D_2 \\ & \text{--- } L_2 : E_0 \succ E_1 , \\ (V_2, -1, E_0, V_1) & x \in D_3 \\ & \text{--- } L_0 : E_1 \succ E_0, l : E_0 \rightleftharpoons -E_0 . \end{cases} \quad (3.33)$$

^{*)} A path is called *simple* if and only if any edge does not appear more than once in the path. If the path from V_2 to V_1 that we seek is not restricted to the simple path $\gamma(x)$ (3.33), there are infinitely many paths that trace the simple path $\gamma(x)$ repeatedly back and forth. The asymptotic evaluation calculated along any of these paths is equal to that calculated along the simple path $\gamma(x)$.

The substitution of (3.33) into the asymptotic formula (2.28) yields

$$\mathcal{I}_{\text{II}}(V_2, V_1; x) = \begin{cases} \mathcal{G}(s_{\text{II},0}(x); x) & x \in D_0 \\ & \text{--- } L_1 : E_0 \succ E_1 , \\ \mathcal{G}(s_{\text{II},0}(x); x) - \mathcal{G}(s_{\text{II},1}(x); x) & x \in D_1 \\ & \text{--- } B : E_0 \rightleftharpoons E_1 , \\ -\mathcal{G}(s_{\text{II},0}(x); x) + \mathcal{G}(s_{\text{II},1}(x); x) & x \in D_2 \\ & \text{--- } L_2 : E_0 \succ E_1 , \\ -\mathcal{G}(s_{\text{II},0}(x); x) & x \in D_3 \\ & \text{--- } L_0 : E_1 \succ E_0, l : E_0 \rightleftharpoons -E_0 , \end{cases} \quad (3.34)$$

where the asymptotic contribution $\mathcal{G}(s_{\text{II}}; x)$ from a saddle point s_{II} is specified by (see (2.26))

$$\mathcal{G}(s_{\text{II}}; x) = \sqrt{\frac{\pi}{|s_{\text{II}}|}} e^{-i\frac{1}{2}\text{Arg } s_{\text{II}}} e^{-\frac{1}{3}s_{\text{II}}^3 + xs_{\text{II}}} \quad \text{with } -\pi \leq \text{Arg } s_{\text{II}} < \pi . \quad (3.35)$$

While the control parameter x moves around once the origin O of base space X , the asymptotic expression $\mathcal{I}_{\text{II}}(V_2, V_1; x)$ (3.34) abruptly changes four times just when x crosses every boundary line between two adjacent atomic domains. Two cases of them are only superficial changes arising from the abrupt change in the correspondence between *name* and *entity*. On the other hand, the other two cases are the real bifurcation phenomena known as the Stokes phenomena.¹²⁾ The former two cases are explained in §3.8.2 and the latter two cases are explained in §3.8.3.

3.8.2. Superficial changes of the asymptotic expression

First, when the control parameter x crosses the branch cut B from the atomic domain D_1 to the atomic domain D_2 , the *entities* of saddle points that are pointed to by the *names* $s_{\text{II},0}(x)$ and $s_{\text{II},1}(x)$ are suddenly exchanged. This sudden exchange in the correspondence between entities and names just cancels the effect of the discontinuity of mathematical *expression* between the 2nd and 3rd lines on the right-hand side of (3.34). Hence, the *value* of asymptotic expression $\mathcal{I}_{\text{II}}(V_2, V_1; x)$ (3.34) is entirely continuous when crossing the branch cut B , although it looks apparently as if it changes discontinuously there.

Next, we discuss the change of asymptotic expression $\mathcal{I}_{\text{II}}(V_2, V_1; x)$ (3.34) while the control parameter x crosses the switching line l from the atomic domain D_3 to the atomic domain D_0 . As shown below, the *value* of asymptotic expression $\mathcal{I}_{\text{II}}(V_2, V_1; x)$ (3.34) is again entirely continuous during this variation of x .

We consider a variation of control parameter $x \in X$ from slightly below the switching line l up to slightly above it. The initial and terminal points of this variation of x are taken as $x = R e^{-i\delta}$ and $x = R e^{+i\delta}$ with some radius $R > 0$ and a positive small value $\delta \gtrsim 0$.

During this variation of $x : R e^{-i\delta} \rightarrow R e^{+i\delta}$, we calculate the change of principal argument function $\text{Arg } s_{\text{II},0}(x) \in [-\pi, \pi)$, which appears in the asymptotic contribution $\mathcal{G}(s_{\text{II},0}(x); x)$ (3.35) from the 0th saddle point $s_{\text{II},0}(x)$. Since the 0th saddle

point $s_{II,0}(x)$ is expressed as (see (3·29))

$$s_{II,0}(x) = \sqrt{|x|} e^{i(\frac{1}{2}\text{Arg } x - \pi)} \quad \text{with } -\pi \leq \text{Arg}(x) < \pi, \quad (3\cdot36)$$

the argument $\arg s_{II,0}(x)$ of it, which is not yet restricted to the range $[-\pi, \pi)$, takes the form

$$\arg s_{II,0}(x) = \frac{1}{2}\text{Arg } x - \pi = \begin{cases} -\frac{\delta}{2} - \pi \pmod{2\pi} & x = R e^{-i\delta}, \\ +\frac{\delta}{2} - \pi \pmod{2\pi} & x = R e^{+i\delta}. \end{cases} \quad (3\cdot37)$$

If the argument $\arg s_{II,0}(x)$ is restricted to the range $[-\pi, \pi)$, then (3·37) becomes

$$\text{Arg } s_{II,0}(x) = \begin{cases} -\frac{\delta}{2} + \pi & x = R e^{-i\delta}, \\ +\frac{\delta}{2} - \pi & x = R e^{+i\delta}. \end{cases} \quad (3\cdot38)$$

This means that the value of argument $\text{Arg } s_{II,0}(x) \in [-\pi, \pi)$ changes discontinuously when crossing the switching line l . This discontinuity is inherited to the discontinuity of phase factor $e^{-i\frac{1}{2}\text{Arg } s_{II,0}(x)}$, which appears in the asymptotic contribution $\mathcal{G}(s_{II,0}(x); x)$ (3·35), as

$$e^{-i\frac{1}{2}\text{Arg } s_{II,0}(x)} = \begin{cases} e^{-i\frac{1}{2}(\pi - \frac{\delta}{2})} \xrightarrow{\delta \rightarrow 0} -i & x = R e^{-i\delta}, \\ e^{+i\frac{1}{2}(\pi - \frac{\delta}{2})} \xrightarrow{\delta \rightarrow 0} +i & x = R e^{+i\delta}. \end{cases} \quad (3\cdot39)$$

Thus, while the control parameter x crosses the switching line l from the atomic domain D_3 to the atomic domain D_0 , the phase factor $e^{-i\frac{1}{2}\text{Arg } s_{II,0}(x)}$ in the asymptotic contribution $\mathcal{G}(s_{II,0}(x); x)$ (3·35) jumps from $-i$ to $+i$. This jump of phase factor $e^{-i\frac{1}{2}\text{Arg } s_{II,0}(x)}$ when crossing the switching line l just compensates the difference of the sign of $\mathcal{G}(s_{II,0}(x); x)$ between the 4th and 1st lines on the right-hand side of (3·34). In conclusion, the *value* of asymptotic expression $\mathcal{I}_{II}(V_2, V_1; x)$ (3·34) is entirely continuous when crossing the switching line l .

Otherwise, if the switching line l is neglected from the base space X , the discontinuity (3·38) of principal argument function $\text{Arg } s_{II,0}(x) \in [-\pi, \pi)$ cannot be compensated anymore and some wrong discontinuity remains in the final asymptotic evaluation $\mathcal{I}_{II}(V_2, V_1; x)$ (3·34).

In existing textbooks on asymptotic analysis that we have inspected,^{2),7)} the asymptotic expression of a target integral is always written down at a time over not the whole control parameter space X but only some part of it such as the upper half-plane of X . In these presentations, a switching line is hidden in the not considered part of X .

3.8.3. Real bifurcation phenomena in the asymptotic evaluation

We finally discuss a real discontinuity that occurs in the *value* of asymptotic expression $\mathcal{I}_{II}(V_2, V_1; x)$ (3·34) of the integral $\mathcal{I}(V_2, V_1; x)$ when the control parameter x crosses a saddle connection line.

First, when x crosses the saddle connection line $L_1 : E_0 \succ E_1$ from the atomic domain D_0 to the atomic domain D_1 , the *sub-dominant term*^{*)} $-\mathcal{G}(s_{II,1}(x); x)$ ap-

^{*)} The *dominant (sub-dominant) term* refers to the contribution $\mathcal{G}(s_{II}(x); x)$ from a dominant (sub-dominant) saddle point $s_{II}(x)$.

pears abruptly in the asymptotic expression (3·34), whereas the *dominant term* $\mathcal{G}(s_{\text{II},0}(x); x)$ remains to exist for both of adjacent atomic domains D_0 and D_1 separated by the saddle connection line L_1 .

Second, when x crosses the saddle connection line $L_2 : E_0 \succ E_1$ from the atomic domain D_2 to the atomic domain D_3 , the sub-dominant term $\mathcal{G}(s_{\text{II},1}(x); x)$ disappears abruptly from the asymptotic expression (3·34), whereas the dominant term $-\mathcal{G}(s_{\text{II},0}(x); x)$ remains to exist for both of adjacent atomic domains D_2 and D_3 separated by the saddle connection line L_2 .

When the control parameter x crosses these saddle connection lines L_1 and L_2 , the value of asymptotic expression $\mathcal{I}_{\text{II}}(V_2, V_1; x)$ discontinuously changes. Such discontinuous phenomenon is well-known as the *Stokes phenomenon*, which was found by Stokes.¹³⁾ In short, the Stokes phenomenon refers to the phenomenon that happens in the asymptotic evaluation $\mathcal{I}_{\text{II}}(V_2, V_1; x)$ in which a *sub-dominant* term appears or disappears abruptly when the control parameter x crosses a saddle connection line L . Such a saddle connection line L is called the *Stokes line*.

Generally speaking, a saddle connection line $L : E^{\text{dominant}} \succ E^{\text{sub-dominant}}$ becomes the Stokes line if and only if the *dominant* edge E^{dominant} is a *contributing* edge (namely it is contained in a path $\gamma(x) \in \Gamma(\mathcal{G}(x); V_i, V_j)$) when x is in adjacent atomic domains D and D' separated by the saddle connection line L .

The other saddle connection phenomenon on the 0th saddle connection line $L_0 : E_1 \succ E_0$ does *not* appear as a Stokes phenomenon in the asymptotic evaluation $\mathcal{I}_{\text{II}}(V_2, V_1; x)$. This is because the dominant edge E_1 on $L_0 : E_1 \succ E_0$ is a *non-contributing* edge when x is in both of adjacent atomic domains D_3 and D_0 separated by L_0 (see (3·33)). Hence, the asymptotic evaluation $\mathcal{I}_{\text{II}}(V_2, V_1; x)$ is entity continuous when x crosses the saddle connection line L_0 from D_3 to D_0 .

§4. Recipe of steepest descent method for an integral involving one control parameter

Through the previous section 3, we have applied the steepest descent method to the Airy function $\text{Ai}(x)$. We now turn our attention to a general integral $\mathcal{I}(V_i, V_j; x) = \int_{V_i \rightarrow V_j} e^{-f(s;x)/\epsilon} ds$ that involves one control parameter $x \in \mathbb{C}$.

In this explanation, we assume that the saddle point equation $f'(s_{\text{II}}(x); x) = 0$ has M solutions $\{s_{\text{II},m}(x) : m = 0, 1, \dots, M-1\}$.*) The mathematical expressions for saddle points $\{s_{\text{II},m}(x)\}$ determine the location of branch cut on the base space X . When the control parameter x crosses a branch cut B , the *entities* of multiple saddle points that are pointed to by some *names* are suddenly exchanged.

The asymptotic evaluation of a general integral $\mathcal{I}(V_i, V_j; x)$ is obtained by the steepest descent method step by step as follows.

*) We consider here straightforwardly the saddle points $\{s_{\text{II},m}(x)\}$ when the control parameter x is on the base space $X \equiv \{x \in \mathbb{C}\}$ not on the system of Riemann sheets $\tilde{X} = \cup_{m=0}^{M-1} X_m$.

(step 1) *Positive stationary phase path and switching line*

For every saddle point $s_{\text{II},m}(x)$, we let $E_m(x)$ denote the positive stationary phase path that passes through it. In our convention, the tangential direction of any positive stationary phase path $E_m(x)$ at $s_{\text{II},m}(x) \in E_m(x)$ points to the *right* half-plane of complex s -plane $\{s \in \mathbb{C}\}$. Namely, the rotation angle $\theta(E_m(x))$ of any positive stationary phase path $E_m(x)$ satisfies

$$-\frac{\pi}{2} < \theta(E_m(x)) \leq \frac{\pi}{2} . \quad (4.1)$$

Hence, the direction of a positive stationary phase path $E_m(x)$ is abruptly reversed when the rotation angle $\theta(E_m(x))$ of it reaches the boundary $\frac{\pi}{2}$ of our convention (4.1). Accordingly, the switching line l , at which this abrupt change occurs, is specified by

$$l = \{x \in X : \theta(E_m(x)) = \frac{\pi}{2}\} . \quad (4.2)$$

We subsequently rewrite this expression (4.2) of switching line l in terms of the saddle point $s_{\text{II},m}(x)$ on the path $E_m(x)$. Since the rotation angle $\theta(E_m(x))$ of $E_m(x)$ is expressed as (see (2.25))

$$\theta(E_m(x)) = -\frac{1}{2} \text{Arg } f''(s_{\text{II},m}(x); x) \quad \text{with } -\pi \leq \text{Arg } f''(s_{\text{II},m}(x); x) < \pi , \quad (4.3)$$

the equation (4.2) is also written as

$$l = \{x \in X : \text{Arg } f''(s_{\text{II},m}(x); x) = -\pi\} . \quad (4.4)$$

This equation means that the end point of a switching line l is a point where $f''(s_{\text{II},m}(x); x)$ vanishes or a point where it diverges.

(step 2) *Saddle connection line*

The location of saddle connection line on the base space X can be calculated numerically from its end point. There are two possible cases as the fate of saddle connection line when it is traced. First, a saddle connection line L may end up at a degenerating point x^{deg} of multiple saddle points. This type of saddle connection line is called the *ordinary saddle connection line*. Second, a saddle connection line L may end up at a crossing point x^{cross} of two other saddle connection lines. This type of saddle connection line, which is called the *new saddle connection line*, is found firstly by Berk, Nevins and Roberts¹⁸⁾ and is examined recently by Kawai and Takei et al. from the viewpoint of exact WKB analysis.²⁰⁾

(step 3) *Atomic domain and connectivity graph*

The base space X is divided by the three types of boundary line (branch cut, switching line and saddle connection line) into multiple atomic domains. For every atomic domain, we pick up one representative point x in it to depict the configuration $\mathcal{C}(x)$ of positive stationary phase paths $\{E_m(x)\}$ on the state variable space $\{s \in \mathbb{C}\}$. The configuration $\mathcal{C}(x)$ for each atomic domain is reduced to the connectivity graph $\mathbf{G}(x) = (\mathbf{V}, \mathbf{E}, \partial^-(x), \partial^+(x))$. The connectivity graph $\mathbf{G}(x)$ takes a constant value as a function of x on each atomic domain.

(step 4) Asymptotic evaluation of integral

The integral $\mathcal{I}(V_i, V_j; x)$ from the initial valley V_i to the terminal valley V_j is asymptotically evaluated along a path $\gamma(x) \in \Gamma(\mathbf{G}(x); V_i, V_j)$ in the graph $\mathbf{G}(x)$ that connects the initial vertex $V_i \in \mathbf{V}$ to the terminal vertex $V_j \in \mathbf{V}$. Along this path $\gamma(x)$, the given integral $\mathcal{I}(V_i, V_j; x)$ is decomposed into elementary integrals $\{\mathcal{I}(E; x) : E \in \mathbf{E}\}$ as (see (2.19))

$$\mathcal{I}(V_i, V_j; x) = \mathcal{I}(\gamma(x); x) = \sum_{l=0}^{|\gamma(x)|-1} \sigma^{(l)}(\gamma(x)) \mathcal{I}(E^{(l)}(\gamma(x)); x) . \tag{4.5}$$

Corresponding to the equalities for exact integrals in (4.5), the equalities for asymptotic evaluations of these integrals hold as (see (2.28))

$$\mathcal{I}_{\text{II}}(V_i, V_j; x) = \mathcal{I}_{\text{II}}(\gamma(x); x) = \sum_{l=0}^{|\gamma(x)|-1} \sigma^{(l)}(\gamma(x)) \mathcal{I}_{\text{II}}(E^{(l)}(\gamma(x)); x) . \tag{4.6}$$

The elementary integral $\mathcal{I}(E_m; x)$ for an edge $E_m \in \mathbf{E}$ is evaluated as $\mathcal{I}_{\text{II}}(E_m; x) = \mathcal{G}(s_{\text{II},m}(x); x)$, where $\mathcal{G}(s_{\text{II},m}(x); x)$ is the asymptotic contribution from the saddle point $s_{\text{II},m}(x)$ on the corresponding stationary phase path $E_m(x)$. The asymptotic contribution $\mathcal{G}(s; x)$ takes the form (see (2.26))

$$\mathcal{G}(s; x) = \sqrt{\frac{2\pi\epsilon}{|f''(s; x)|}} e^{-i\frac{1}{2}\text{Arg } f''(s; x)} e^{-f(s; x)/\epsilon} \quad \text{with } -\pi \leq \text{Arg } f''(s; x) < \pi . \tag{4.7}$$

§5. Application to the BNR integral

In this section 5, our recipe shown in the previous section 4 is used to derive the asymptotic expression of the BNR integral,^{18),19)} which is a special case of the cusp diffraction integral.¹⁷⁾ The BNR integral is well-known as an integral that exhibits the *new Stokes phenomenon*^{18),20)} and thereby is a nice example to utilize our recipe of steepest descent method.

The BNR equation in our convention is $\left(\frac{d^3}{dx^3} + 3\frac{d}{dx} - i2x\right)\Phi(x) = 0$. A solution of this differential equation is called the *BNR integral*

$$\Phi(x) = \int_{V_2 \rightarrow V_3} e^{-f(s;x)} ds \equiv \mathcal{I}(V_2, V_3; x) , \tag{5.1}$$

where the analytic function $f(s; x)$ of the state variable $s \in \mathbb{C}$ and the control parameter $x \in \mathbb{C}$ is given by

$$f(s; x) = -i \left(\frac{1}{8}s^4 + \frac{3}{4}s^2 \right) - xs . \tag{5.2}$$

The integrand $e^{-f(s;x)} = e^{i(s^4/8+3s^2/4)+xs}$ in the BNR integral $\Phi(x)$ has four valleys

$$V_n = \left\{ s \in S : |s| \rightarrow \infty \wedge \frac{n\pi}{2} < \arg s < \frac{n\pi}{2} + \frac{\pi}{4} \right\} \quad \text{for } n = 0, 1, 2, 3 . \tag{5.3}$$

(step 0) Saddle point and branch cut

The saddle point equation (SE) : $f'(s_{II}(x); x) = 0$ becomes

$$\{s_{II}(x)\}^3 + 3s_{II}(x) - i2x = 0 . \tag{5.4}$$

This cubic equation (5.4) is solved as

$$s_{II,m}(x) = -i \left(\omega^m u(x) + \frac{1}{\omega^m u(x)} \right) \quad \text{for } m = 0, 1, 2 , \tag{5.5}$$

where $\omega = e^{i2\pi/3}$ and $u(x) \equiv \left\{ x + \{x^2 - 1\}^{1/2} \right\}^{1/3}$. Here, $\{z\}^{1/n}$ for $z \in \mathbb{C}$ and $n \in \mathbb{N}$ obeys the convention $\{z\}^{1/n} \equiv |z|^{1/n} e^{i \arg(z)/n}$ with $0 \leq \arg(z) < 2\pi$.

The discriminant of the cubic equation (5.4) with respect to $s_{II}(x)$ vanishes at the two points $x_0^{\text{deg}} = 1$ and $x_1^{\text{deg}} = -1$. These degenerating points are the fold-type and hence two saddle points collide there as

$$s_{II,1}(x_0^{\text{deg}}) = s_{II,2}(x_0^{\text{deg}}) \quad \text{and} \quad s_{II,0}(x_1^{\text{deg}}) = s_{II,2}(x_1^{\text{deg}}) . \tag{5.6}$$

In addition, the discriminant of the cubic equation (5.4) with respect to $1/s_{II}(x)$ vanishes at the point $x_2^{\text{deg}} = \infty$. This degenerating point x_2^{deg} is the cusp-type and hence three saddle points collide there.

The location of branch cut on the base space $X \equiv \{x \in \mathbb{C}\}$ is determined from the mathematical expression (5.5) of saddle points $\{s_{II,m}(x)\}$. As shown in Fig. 10, there are two branch cuts

$$\begin{aligned} B_0 &= \{x \in X : |x| \geq 1 \wedge \arg x = 0\} , \\ B_1 &= \{x \in X : |x| \geq 1 \wedge \arg x = \pi\} . \end{aligned} \tag{5.7}$$

Both branch cuts B_0 and B_1 end up at the cusp degenerating point $x_2^{\text{deg}} = \infty$. When x crosses the branch cut B_0 , the entities of two saddle points that are pointed to by the names $s_{II,1}(x)$ and $s_{II,2}(x)$ are suddenly exchanged, i.e. $B_0 : s_{II,1} \rightleftharpoons s_{II,2}$. Similarly, $B_1 : s_{II,0} \rightleftharpoons s_{II,2}$ holds.

(step 1) Positive stationary phase path and switching line

This step determines the location of switching line, on which some saddle point $s_{II,m}(x)$ satisfies $\text{Arg } f''(s_{II,m}(x); x) = -\pi$ (4.4). As shown in Fig. 10, there are two switching lines

$$\begin{aligned} l_0 &= \{x \in X : \text{Arg } f''(s_{II,0}(x); x) = -\pi\} , \\ l_1 &= \{x \in X : \text{Arg } f''(s_{II,1}(x); x) = -\pi\} . \end{aligned} \tag{5.8}$$

When x crosses the switching line l_0 , the direction of the 0th positive stationary phase path $E_0(x)$ is suddenly reversed, i.e. $l_0 : E_0 \rightleftharpoons -E_0$. Similarly, $l_1 : E_1 \rightleftharpoons -E_1$ holds.

(step 2) Saddle connection line

Since the degenerating points x_0^{deg} and x_1^{deg} are fold-type, three saddle connection lines emanate from each fold degenerating point x_d^{deg} ($d = 0, 1$). The three saddle

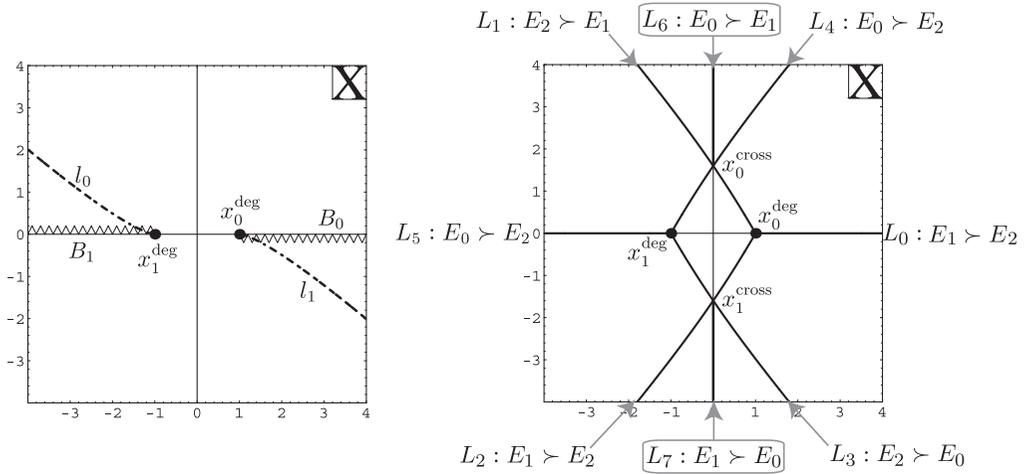


Fig. 10. **(Left)** Location of branch cut and switching line on the base space X . **(Right)** Location of saddle connection line on X . The collisions of multiple saddle points occur at $x_0^{\text{deg}} = 1$ and $x_1^{\text{deg}} = -1$. The crossing points of multiple saddle connection lines are $x_0^{\text{cross}} = i\frac{1}{2}\sqrt{5 + 3\sqrt{3}}$ and $x_1^{\text{cross}} = -i\frac{1}{2}\sqrt{5 + 3\sqrt{3}}$. The new saddle connection lines L_6 and L_7 go away respectively from these crossing points x_0^{cross} and x_1^{cross} .

connection lines emanating from the 0th degenerating point x_0^{deg} are named

$$L_0 : E_1 \succ E_2, L_1 : E_2 \succ E_1 \quad \text{and} \quad L_2 : E_1 \succ E_2. \tag{5.9}$$

Similarly, the three saddle connection lines emanating from the 1st degenerating point x_1^{deg} are named

$$L_3 : E_2 \succ E_0, L_4 : E_0 \succ E_2 \quad \text{and} \quad L_5 : E_0 \succ E_2. \tag{5.10}$$

As seen from the right figure in Fig. 10, two saddle connection lines L_1 and L_4 cross at the point x_0^{cross} , on which the dominance relation $E_0 \succ E_2 \succ E_1$ holds. Similarly, two saddle connection line L_2 and L_3 cross at the point x_1^{cross} , on which the dominance relation $E_1 \succ E_2 \succ E_0$ holds. As shown in Refs. 18) and 20) two *new saddle connection lines*

$$L_6 : E_0 \succ E_1 \quad \text{and} \quad L_7 : E_1 \succ E_0 \tag{5.11}$$

go away respectively from these crossing points x_0^{cross} and x_1^{cross} .

(step 3) Atomic domain and connectivity graph

By all saddle connection lines $\{L_n : n = 0, 1, 2\}$, all branch cuts $\{B_0, B_1\}$ and all switching lines $\{l_0, l_1\}$, the base space X is divided into 11 atomic domains $\{D_a : a = 0, 1, \dots, 10\}$. Figure 11 depicts the configuration $\mathcal{C}(x)$ of positive stationary phase paths for a representative point x in each atomic domain. In Fig. 12, each configuration $\mathcal{C}(x)$ is reduced to the connectivity graph $G(x)$.

(step 4) Asymptotic evaluation of integral

The BNR integral $\Phi(x) = \mathcal{I}(V_2, V_3; x)$ is evaluated asymptotically along a path $\gamma(x)$ in the graph G that connects the initial vertex $V_2 \in \mathbb{V}$ to the terminal vertex

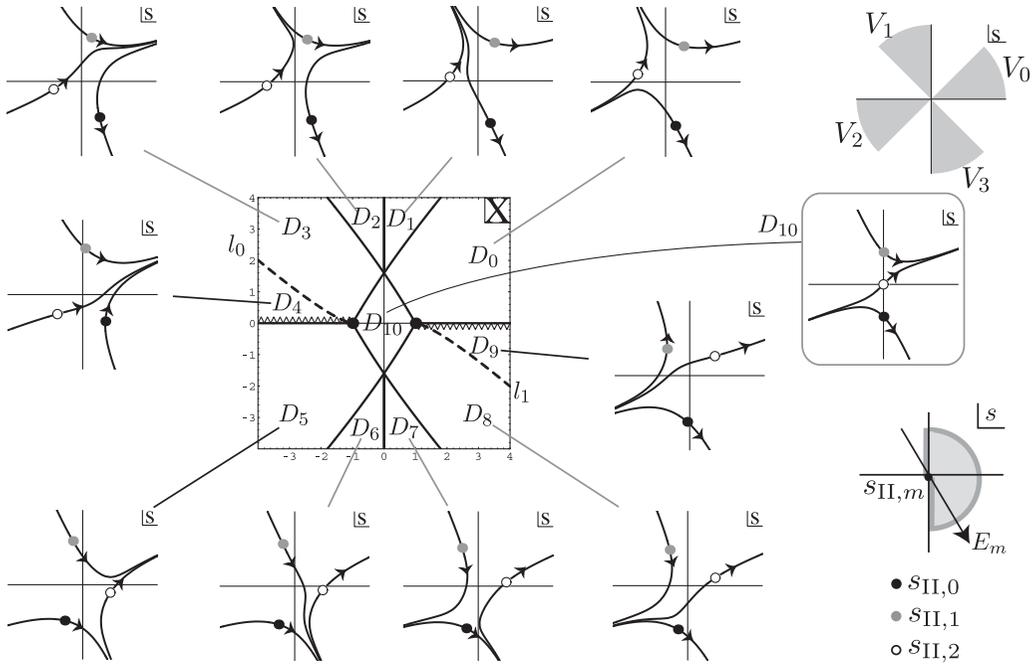


Fig. 11. Configuration $\mathcal{C}(x)$ of positive stationary phase paths for a representative point x in each atomic domain. The black dot, the gray dot and the circle in $\mathcal{C}(x)$ represent respectively the 0th, 1st and 2nd saddle points. The arrow (\rightarrow) attached to the saddle point $s_{II,m}(x)$ represents the direction of the path $E_m(x)$.

$V_3 \in V$. The simple path $\gamma(x)$ from V_2 to V_3 is easily read from Fig. 12 as

$$\gamma(x) = \begin{cases} (V_2, +1, E_0, V_3) & x \in \bar{D} , \\ (V_2, +1, E_2, V_1, +1, E_0, V_3) & x \in D_1 , \\ (V_2, +1, E_2, V_1, +1, E_1, V_0, +1, E_0, V_3) & x \in D_2 , \\ (V_2, +1, E_2, V_1, +1, E_0, V_3) & x \in D_3 , \\ (V_2, +1, E_2, V_0, -1, E_0, V_3) & x \in D_4 . \end{cases} \quad (5.12)$$

The domain \bar{D} that appears at the 1st line in (5.12) is defined by $\bar{D} \equiv D_0 \cap D_5 \cap D_6 \cap D_7 \cap D_8 \cap D_9 \cap D_{10}$.

Next, by substituting (5.12) into the asymptotic formula (4.6), the asymptotic expression $\Phi_{II}(x) = \mathcal{I}_{II}(V_2, V_3; x)$ of the BNR integral $\Phi(x) = \mathcal{I}(V_2, V_3; x)$ is derived as

$$\Phi_{II}(x) = \begin{cases} \mathcal{G}(s_{II,0}(x); x) & x \in \bar{D} , \\ \mathcal{G}(s_{II,2}(x); x) + \mathcal{G}(s_{II,0}(x); x) & x \in D_1 , \\ \mathcal{G}(s_{II,2}(x); x) + \mathcal{G}(s_{II,1}(x); x) + \mathcal{G}(s_{II,0}(x); x) & x \in D_2 , \\ \mathcal{G}(s_{II,2}(x); x) + \mathcal{G}(s_{II,0}(x); x) & x \in D_3 , \\ \mathcal{G}(s_{II,2}(x); x) - \mathcal{G}(s_{II,0}(x); x) & x \in D_4 , \end{cases} \quad (5.13)$$

where the asymptotic contribution $\mathcal{G}(s_{II,m}(x); x)$ from a saddle point $s_{II,m}(x)$ is given

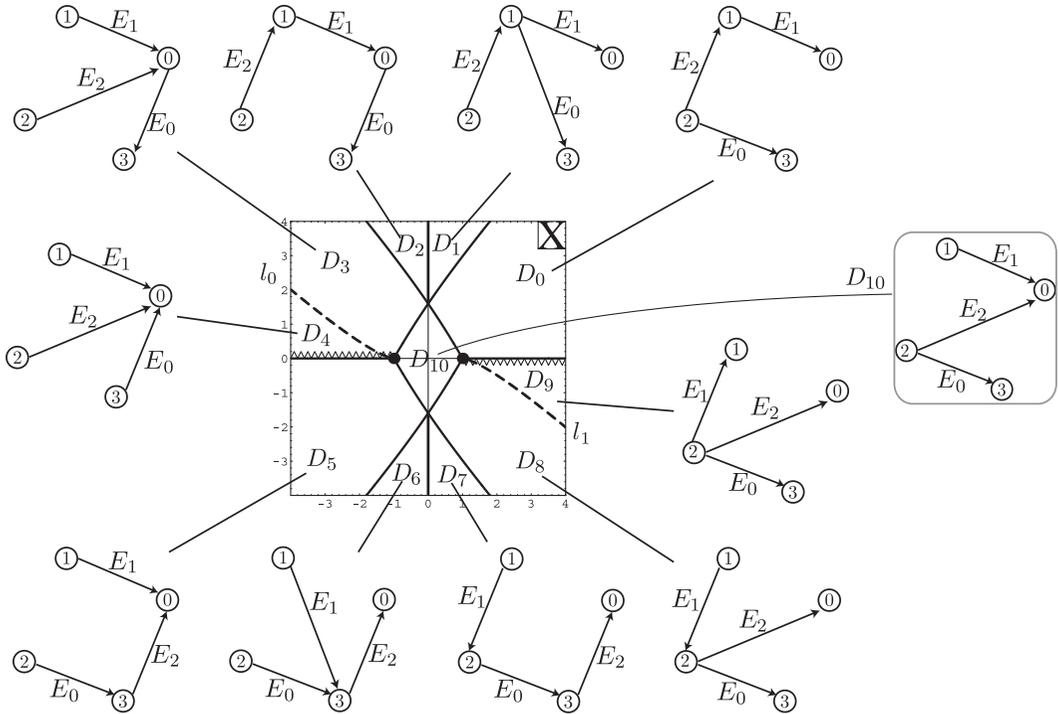


Fig. 12. Connectivity graph $G(x)$ for each atomic domain. In each graph, the circled number n represents a vertex $V_n \in \mathbf{V}$ and the arrow (\longrightarrow) represents an edge $E_m \in \mathbf{E}$.

by (4.7).

We comment on a superficial change of asymptotic expression $\Phi_{II}(x)$ when the control parameter x crosses the switching line l_0 from the atomic domain D_3 to the atomic domain D_4 . The change of sign of the asymptotic contribution $\mathcal{G}(s_{II,0}(x); x)$ from $+$ (4th line in (5.13)) to $-$ (5th line in it) just compensates the abrupt change of the phase factor $e^{-i\frac{1}{2}\text{Arg} f''(s_{II,0}(x); x)}$ in $\mathcal{G}(s_{II,0}(x); x)$ when x crosses the switching line l_0 (5.8).

The final asymptotic expression $\Phi_{II}(x)$ (5.13) obtained in this manner has no obscureness concerning the multi-valuedness that comes from both of the saddle point $s_{II}(x)$ and the square root $\sqrt{\frac{2\pi\epsilon}{f''(s_{II}(x); x)}}$ in the local asymptotic evaluation. Accordingly, this asymptotic expression $\Phi_{II}(x)$ can be translated mechanically to a computer program that calculates the numerical values of the asymptotic expression $\Phi_{II}(x)$ of the BNR integral $\Phi(x)$. In Fig. 13, the asymptotic evaluation $\Phi_{II}(x)$ (5.13) is plotted for real values of x ; the exact BNR integral $\Phi(x)$ is also plotted there for comparison.

In this numerical calculation of the asymptotic evaluation $\Phi_{II}(x)$, we focus our attention to the point whether this asymptotic evaluation $\Phi_{II}(x)$ diverges at the degenerating points $x_1^{\text{deg}} = -1$ and $x_0^{\text{deg}} = 1$. As seen from Fig. 13, the asymptotic evaluation $\Phi_{II}(x)$ diverges at the 1st degenerating point $x_1^{\text{deg}} = -1$, whereas it does not diverge at the 0th degenerating point $x_0^{\text{deg}} = 1$. We subsequently explain why

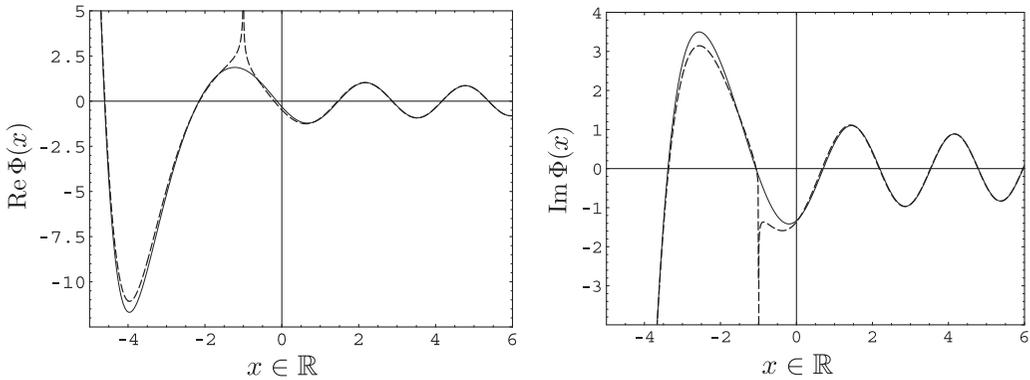


Fig. 13. The asymptotic evaluation $\Phi_{\text{II}}(x)$ (dashed line) of the BNR integral $\Phi(x)$ (solid line) for real values of x . The left and right figures show the real and imaginary parts of them. The asymptotic evaluation $\Phi_{\text{II}}(x)$ diverges at the 1st degenerating point $x_1^{\text{deg}} = -1$ but it does not diverge at the 0th one $x_0^{\text{deg}} = 1$.

these different behaviors at the degenerating points $x_1^{\text{deg}} = -1$ and $x_0^{\text{deg}} = 1$ appear in the asymptotic evaluation $\Phi_{\text{II}}(x)$.

According to a general theory of steepest descent method, the asymptotic evaluation $\mathcal{I}_{\text{II}}(V_i, V_j; x)$ of a given integral $\mathcal{I}(V_i, V_j; x)$ diverges at a degenerating point x^{deg} of multiple saddle points if and only if either of these saddle points is *contributing* to the asymptotic evaluation $\mathcal{I}_{\text{II}}(V_i, V_j; x)$ whenever the control parameter x is around the degenerating point x^{deg} . In the present application to the BNR integral $\Phi(x)$, the asymptotic evaluation $\Phi_{\text{II}}(x)$ diverges at the 1st degenerating point $x_1^{\text{deg}} = -1$, since the 0th saddle point $s_{\text{II},0}(x)$, which collides with the 2nd saddle point $s_{\text{II},2}(x)$ at $x_1^{\text{deg}} = -1$, is contributing to the asymptotic evaluation $\Phi_{\text{II}}(x)$ whenever x is around x_1^{deg} . On the other hand, at the 0th degenerating point $x_0^{\text{deg}} = 1$, the asymptotic evaluation $\Phi_{\text{II}}(x)$ does not diverges, since both of the saddle points $s_{\text{II},1}(x)$ and $s_{\text{II},2}(x)$, which degenerate at the 0th degenerating point $x_0^{\text{deg}} = 1$, are non-contributing to the asymptotic evaluation $\Phi_{\text{II}}(x)$ whenever x is around x_0^{deg} . This situation is said that “the collision of two saddle points $s_{\text{II},1}(x)$ and $s_{\text{II},2}(x)$ just at $x = x_0^{\text{deg}}$ is *irrelevant* to the asymptotic evaluation of the BNR integral $\Phi(x)$ ”.

In this and subsequent papers,^{11),21)} the degenerating point x^{deg} of multiple saddle points of a given integral $\mathcal{I}(V_i, V_j; x)$ is said to be a *caustic* of the integral $\mathcal{I}(V_i, V_j; x)$ if and only if the asymptotic evaluation of $\mathcal{I}(V_i, V_j; x)$ diverges at $x = x^{\text{deg}}$. Namely, any caustic is always a degenerating point but every degenerating point does not need to be a caustic. In the present application, the 1st degenerating point $x_1^{\text{deg}} = -1$ is a caustic of the BNR integral $\Phi(x)$, whereas the 0th degenerating point $x_0^{\text{deg}} = 1$ is not a caustic of it.

§6. Future work

At present, we have treated some integral $\mathcal{I}(V_i, V_j; x)$ involving just one complex control parameter $x \in \mathbb{C}$. One of our remaining problems is to extend the theory presented in this paper to treat an integral $\mathcal{I}(V_i, V_j; \mathbf{x})$ involving multiple complex control parameters $\mathbf{x} \equiv (x_0, x_1, \dots, x_{K-1}) \in \mathbb{C}^K$. This extension requires us to investigate the global bifurcation phenomena of such an integral in some higher dimensional space of complex control parameters. This may be a difficult investigation. When this extension is achieved, the global bifurcation phenomena of the diffraction integrals,¹⁷⁾ such as the cusp diffraction integral, the swallow-tail integral and so on, are understood thoroughly over the whole of complex parameter space.

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Appendix A

— Path in Graph Theory —

This appendix explains the *path* in graph theory.^{22),23)} For a graph $G = (V, E, \partial^-, \partial^+)$, if a sequence γ of vertices $\{v^{(l)} \in V\}$ and edges $\{e^{(l)} \in E\}$ with signs $\{\sigma^{(l)} \in \{-1, +1\}\}$ of the form

$$\gamma = (v^{(0)}, \sigma^{(0)}, e^{(0)}, v^{(1)}, \sigma^{(1)}, e^{(1)}, \dots, \sigma^{(L-1)}, e^{(L-1)}, v^{(L)}) \quad (\text{A}\cdot 1)$$

satisfies the connectivity condition

$$(\partial^- e^{(l)}, \partial^+ e^{(l)}) = \begin{cases} (v^{(l)}, v^{(l+1)}) & \sigma^{(l)} = +1 \\ (v^{(l+1)}, v^{(l)}) & \sigma^{(l)} = -1 \end{cases} \quad \text{for } l = 0, 1, \dots, L-1, \quad (\text{A}\cdot 2)$$

then the sequence γ is called a path in the graph G . The positive integer $L \in \mathbb{N}$ means the length of the path γ and is denoted by $|\gamma|$. A path is called *simple* if and only if any edge does not appear more than once in the path.

The set of paths in a graph G whose initial and terminal vertices are v and v' respectively is denoted by $\Gamma(G; v, v')$. The set of all paths in the graph G is denoted by $\Gamma(G)$, i.e. $\Gamma(G) = \bigcup_{v, v' \in V} \Gamma(G; v, v')$. The l -th vertex $v^{(l)}$ of a path γ is denoted by $v^{(l)}(\gamma)$ for $l = 0, 1, \dots, |\gamma|$. Similarly, the l -th sign $\sigma^{(l)}$ and the l -th edge $e^{(l)}$ of a path γ are denoted respectively by $\sigma^{(l)}(\gamma)$ and $e^{(l)}(\gamma)$ for $l = 0, 1, \dots, |\gamma| - 1$.

When a function $\eta(\cdot)$ from the set of edges E to some range R is given, this function $\eta : E \rightarrow R$ induces naturally another function $\tilde{\eta} : \Gamma(G) \rightarrow R$ from the set of paths $\Gamma(G)$ to the range R by the equation

$$\tilde{\eta}(\gamma) \equiv \sum_{l=0}^{|\gamma|-1} \sigma^{(l)}(\gamma) \eta(e^{(l)}(\gamma)) \quad \text{for } \gamma \in \Gamma(G) . \quad (\text{A}\cdot\text{3})$$

Frequently, the induced function $\tilde{\eta}(\cdot)$ is written as $\eta(\cdot)$ for notational simplicity.

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