

## Phantom Boundary Crossing and Anomalous Growth Index of Fluctuations in Viable $f(R)$ Models of Cosmic Acceleration

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The evolution of a background space-time metric and subhorizon matter density perturbations in the Universe is numerically analyzed in viable  $f(R)$  models of present dark energy and cosmic acceleration. It is found that viable models generically exhibit recent crossing of the phantom boundary  $w_{\text{DE}} = -1$ . Furthermore, it is shown that, as a consequence of the anomalous growth of density perturbations during the end of the matter-dominated stage, their growth index evolves nonmonotonically with time and may even become negative temporarily.

Subject Index: 451, 453, 460

### §1. Introduction

The physical origin of the dark energy (DE) that is responsible for an accelerated expansion of the current Universe is one of the largest mysteries not only in cosmology but also in fundamental physics.<sup>1)</sup> Although the standard spatially flat  $\Lambda$ -Cold-Dark-Matter ( $\Lambda$ CDM) model is consistent with all types of current observational data,<sup>2)</sup> some tentative deviations from it have been reported recently,<sup>3),4)</sup> which, if proven to be not due to systematic and other errors, may eventually rule out an exact cosmological constant. Furthermore, in the  $\Lambda$ CDM model, the cosmological term is regarded as a new fundamental constant whose observed value is much smaller than any other energy scale known in physics. Thus, its understanding in fundamental physics is lacking today, although some nonperturbative effects may generate such a small quantity.<sup>5)</sup> On the other hand, we know that “primordial DE”, which is responsible for inflation in the early universe,<sup>6)–8)</sup> is not identical to the cosmological constant, in particular, it is not stable and eternal. Hence, it is natural to seek also nonstationary models of the current DE.

Among them,  $f(R)$  gravity that modifies and generalizes the Einstein gravity by incorporating a new phenomenological function of the Ricci scalar  $R$ ,  $f(R)$ , provides a self-consistent and nontrivial alternative to the  $\Lambda$ CDM model, see e.g., Ref. 9) for a recent review. This theory is a special class of the scalar-tensor theory of gravity with the vanishing Brans-Dicke parameter  $\omega_{BD}$ .<sup>10),11)</sup> It contains a new scalar degree of

freedom dubbed “scalaron” in Ref. 6), thus, it is a *nonperturbative* generalization of the Einstein gravity.

This additional degree of freedom imposes a number of conditions on viable functional forms of  $f(R)$ . In particular, in order to have the correct Newtonian limit for  $R \gg R_0 \equiv R(t_0) \sim H_0^2$ , where  $t_0$  is the present moment and  $H_0$  is the Hubble constant, as well as the standard matter-dominated stage with the scale factor behaviour  $a(t) \propto t^{2/3}$  driven by cold dark matter and baryons, the following conditions should be fulfilled:

$$|f(R) - R| \ll R, \quad |f'(R) - 1| \ll 1, \quad Rf''(R) \ll 1, \quad R \gg R_0, \quad (1.1)$$

where the prime denotes the derivative with respect to the argument  $R$ . In addition, the stability condition  $f''(R) > 0$  has to be satisfied, which guarantees that the standard matter-dominated Friedmann stage remains an attractor with respect to an open set of neighboring isotropic cosmological solutions in  $f(R)$  gravity. In quantum language, this condition means that a scalaron is not a tachyon. Note that the other stability condition,  $f'(R) > 0$ , which means that gravity is attractive and graviton is not a ghost, is automatically fulfilled in this regime. Specific functional forms that satisfy all these conditions have been proposed in Refs. 12)–14), and much work has been carried out on their cosmological consequences.

In the previous paper,<sup>15)</sup> we calculated the evolution of matter density fluctuations in viable  $f(R)$  models<sup>12),14)</sup> in the limiting case  $R \gg R_0$  during the matter-dominated stage and found an analytic expression for them. In this paper, we extend the previous analysis and perform numerical calculations of the evolution of both background space-time and density fluctuations for the particular  $f(R)$  model described in Ref. 14) without such restriction on  $R$ . As a result, we have found the phantom boundary crossing at an intermediate redshift  $z \lesssim 1$  for the background space-time metric and an anomalous behaviour of the growth index of fluctuations.

The rest of the paper is organized as follows. In §2, we introduce evolution equations for the homogeneous and isotropic background and present results of numerical integration. In §3, we report on numerical solutions for the evolution of density fluctuations and other observables. Section 4 is devoted to conclusions and discussion.

## §2. Evolution of the background Universe

We adopt the following action with a four-parameter family of  $f(R)$  models:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R) + S_m, \quad (2.1)$$

$$f(R) = R + \lambda R_s \left[ \left( 1 + \frac{R^2}{R_s^2} \right)^{-n} - 1 \right] + \frac{R^2}{6M^2}, \quad (2.2)$$

where  $n$ ,  $\lambda$ ,  $R_s$ , and  $M$  are model parameters and  $S_m$  is the action of the matter content that is assumed to be minimally coupled to gravity (thus, the action (2.1) is written in the Jordan frame). This is the model described in Ref. 14) modified

by the last term in (2.2) borrowed from the inflationary model described in Ref. 6). This term is introduced for several purposes associated with the high-curvature behaviour of the theory. One of them, as explained in Ref. 14), is to avoid excessive growth of the scalaron mass,  $m_s^2 = 1/3f''(R)$  in the regime (1.1), towards the early Universe,  $t \rightarrow 0$ . The other one is to remove the additional and undesirable “Big Boost” singularity, which can arise in the original models,<sup>12)–14)</sup> as was shown in Ref. 16) (see Refs. 15), 17) and 18) for more discussion on this point). The value of  $M$  should be sufficiently large so as not to destroy the standard cosmology of the present and early Universe. In particular, the values of  $M$  considered in Refs. 19) and 20) are not sufficiently high for this purpose, because  $M$  should not be smaller than the Hubble parameter  $H(t)$  during the  $N \sim 60$  last e-folds of inflation in the early Universe to avoid the overproduction of relic scalarons, as well as to solve other cosmological problems. In fact, if we take  $M \approx 3 \times 10^{13}$  GeV, the scalaron itself can act as an inflaton<sup>6)</sup> and generate primordial scalar (adiabatic) and tensor perturbations<sup>21), 22)</sup> with the amplitudes and slopes of their power spectra in agreement with all observational data available today. Note, however, that as shown in Ref. 18), such a “unified” model describing both primordial DE driving inflation in the early Universe and present DE driving recent acceleration of the Universe in the scope of  $f(R)$  gravity leads to slightly different predictions for parameters of the primordial perturbation spectra, as compared with the purely inflationary model with  $\lambda R_s = 0$ , owing to a change in the number of observable e-folds of inflation  $N$  caused by a different evolution of the Universe during the generation and heating of usual matter after inflation. Furthermore, in this unified model, the term in the square brackets in (2.2) should be modified for  $|R| < R_0$  in such a way as to ensure also the fulfillment of the stability condition  $f''(R) > 0$  in this region.

Thus, we take this value of  $M$  and assume that the evolution of the Universe is identical to that in the standard  $\Lambda$ CDM model at high redshifts without any relic scalaron oscillations. Then, the  $R^2/6M^2$  term is totally negligible in the epoch we are concerned here. Therefore, we do not include its contribution below.

We can express field equations derived from the action in the following Einsteinian form,

$$R_\nu^\mu - \frac{1}{2}\delta_\nu^\mu R = -8\pi G \left( T_{\nu(m)}^\mu + T_{\nu(\text{DE})}^\mu \right), \quad (2.3)$$

where

$$8\pi G T_{\nu(\text{DE})}^\mu \equiv \mathcal{F}'(R)R_\nu^\mu - \frac{1}{2}\mathcal{F}(R)\delta_\nu^\mu + (\nabla^\mu \nabla_\nu - \delta_\nu^\mu \square)\mathcal{F}'(R), \quad \mathcal{F}(R) \equiv f(R) - R \quad (2.4)$$

(the sign conventions here are the same as in Ref. 14)). Working in the spatially flat Friedmann-Robertson-Walker (FRW) space-time with the scale factor  $a(t)$ , we find

$$3H^2 = 8\pi G\rho - 3\mathcal{F}'H^2 + \frac{1}{2}(\mathcal{F}'R - \mathcal{F}) - 3H\dot{\mathcal{F}}', \quad (2.5)$$

$$2\dot{H} = -8\pi G\rho - 2\mathcal{F}'\dot{H} - \ddot{\mathcal{F}}' + H\dot{\mathcal{F}}', \quad (2.6)$$

where  $H$  is the Hubble parameter and  $\rho$  is the energy density of the material content, which we assume to consist of nonrelativistic matter.

From (2.4), the effective energy density and pressure of dark energy can be expressed as

$$8\pi G\rho_{\text{DE}} = \frac{1}{2}(\mathcal{F}'R - \mathcal{F}) - 3H^2\mathcal{F}' - 3H\dot{\mathcal{F}}' = -3H\dot{R}\mathcal{F}'' + 3(H^2 + \dot{H})\mathcal{F}' - \frac{1}{2}\mathcal{F}, \tag{2.7}$$

$$8\pi G(\rho_{\text{DE}} + P_{\text{DE}}) = 2\dot{H}\mathcal{F}' - H\dot{\mathcal{F}}' + \ddot{\mathcal{F}}', \tag{2.8}$$

respectively, where  $R = 12H^2 + 6\dot{H}$ . We define the DE equation of state parameter  $w_{\text{DE}}$  by the ratio  $w_{\text{DE}} \equiv P_{\text{DE}}/\rho_{\text{DE}}$ .

With the appropriate initial condition after the cosmic inflation mentioned above,  $\mathcal{F}$  takes an asymptotically constant value  $\mathcal{F} = -\lambda R_s$  at high redshift (apart from the  $R^2/6M^2$  term that we neglect here). In this regime, the evolution of the Universe is the same as that obtained from the Einstein action with a cosmological constant  $\Lambda(\infty) = \lambda R_s/2$ . The scale factor therefore evolves as

$$a = a_i \left( \frac{16\pi G\rho_i}{\lambda R_s} \right)^{\frac{1}{3}} \sinh^{\frac{2}{3}} \left( \sqrt{\frac{3\lambda R_s}{8}} t \right) \cong a_i \left( \frac{t}{t_i} \right)^{\frac{2}{3}}, \tag{2.9}$$

where the suffix  $i$  denotes quantities at an initial time  $t = t_i$ .

The time dependence of  $\rho_{\text{DE}}$  is mainly governed by the first term in the right-most expression of (2.7) initially. Since  $\dot{R} < 0$  and  $\mathcal{F}'' > 0$  for stability, this means that the effective energy density of dark energy *increases* with time in this regime. Therefore, DE exhibits the phantom behaviour,  $w_{\text{DE}} < -1$ , during the matter-dominated stage with  $z > 1$  (more exactly, it occurs for all  $n > (\sqrt{73}-7)/24 \approx 0.064$ ). However, this lasts only temporarily because the late-time asymptotic de Sitter stage has an effective cosmological constant smaller than  $\Lambda(\infty)$ . Thus,  $\rho_{\text{DE}}$  stops growing after the end of the matter-dominated stage and begins to decrease.

Indeed, as shown in Ref. 14), the late-time asymptotic de Sitter solution has a curvature  $R \equiv R_1 \equiv x_1 R_s$ , where  $x_1$  is the maximal solution of the equation,

$$\lambda = \frac{x(1+x^2)^{n+1}}{2[(1+x^2)^{n+1} - 1 - (n+1)x^2]}. \tag{2.10}$$

It satisfies the inequality  $x_1 < 2\lambda$ , so that  $\Lambda(R_1) = R_1/4 < \Lambda(\infty)$ . These inequalities are saturated in the limit  $n \gg 1$  for fixed  $x_1$ , or  $x_1 \gg 1$  for fixed  $n$ . In these cases, cosmic evolution is indistinguishable from the standard  $\Lambda$ CDM model.

Thus, this model naturally realizes the crossing of the phantom boundary  $w_{\text{DE}} = -1$  in a recent epoch. Note that the phantom behaviour of DE is generic in its models based on the scalar-tensor gravity,<sup>23)</sup> which includes the  $f(R)$  theory. Here, we see that it is realized in all the simplest stable  $f(R)$  models of the present DE.

The stability condition of this future de Sitter solution,<sup>24)</sup>  $f'(R_1) > R_1 f''(R_1)$ , imposes the following constraint on  $x_1$ ,

$$(1+x_1^2)^{n+2} > 1 + (n+2)x_1^2 + (n+1)(2n+1)x_1^4, \tag{2.11}$$

which is stronger than any other constraint discussed above. For each  $n$ , we can find  $x_1$ , which marginally satisfies (2.11) and gives the minimal allowed value of

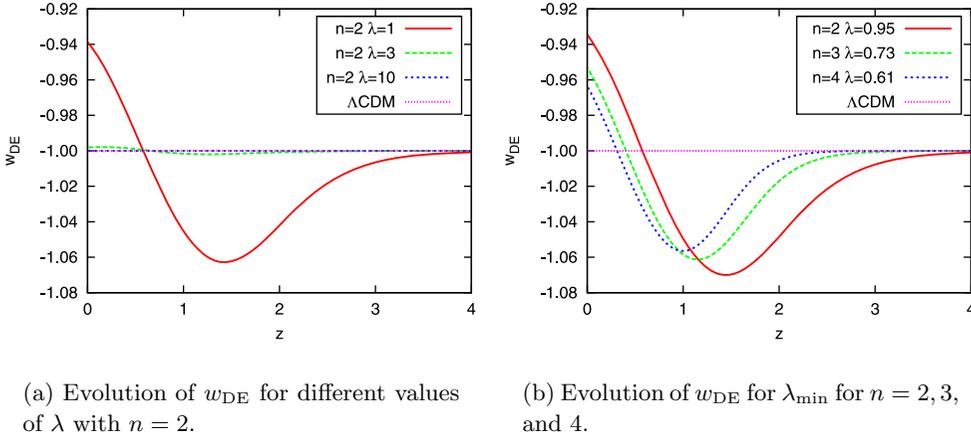


Fig. 1. Evolution of the equation-of-state parameter of effective dark energy.

$\lambda$ . Numerically we find  $(n, x_{1\text{min}}, \lambda_{\text{min}}) = (2, 1.267, 0.9440)$ ,  $(3, 1.041, 0.7259)$ , and  $(4, 0.9032, 0.6081)$  for each  $n$ , respectively (if  $n = 2$ , the analytic expression for  $x_{1\text{min}}$  is  $x_{1\text{min}}^2 = \sqrt{13} - 2$ ). For comparison, the analytic results for  $n = 1$  are  $x_{1\text{min}} = \sqrt{3} \approx 1.732$ ,  $\lambda_{\text{min}} = 8/(3\sqrt{3}) \approx 1.540$ .

We numerically solve evolution equation (2.6) using (2.5) to check the numerical accuracy, taking  $t_i$  at the epoch when the matter density parameter took  $\Omega_i = 16\pi G\rho_i/(16\pi G\rho_i + \lambda R_s) = 0.998$ . We determine the current epoch with the requirement that the value of  $\Omega$  takes the observed central value  $\Omega_0 = 0.27$  and  $R_s$  is fixed so that the current Hubble parameter  $H_0 = 72$  km/s/Mpc is reproduced. We find that the ratio  $R_s/H_0^2$  is well fitted by a simple power-law  $R_s/H_0^2 = c_n \lambda^{-p_n}$  with  $(n, c_n, p_n) = (2, 4.16, 0.953)$ ,  $(3, 4.12, 0.837)$ , and  $(4, 4.74, 0.702)$ , respectively, whereas in the  $\Lambda\text{CDM}$  limit, it would behave as  $R_s/H_0^2 = 6(1 - \Omega_0)/\lambda \simeq 4.38\lambda^{-1}$ .

Figure 1 depicts the evolution of  $w_{\text{DE}}$  as a function of redshift  $z$  where phantom crossing is manifest. As expected, it approaches  $w_{\text{DE}} = -1 = \text{constant}$  as we increase  $\lambda$  for fixed  $n$ . For minimal allowed values of  $\lambda$ , deviations from  $w_{\text{DE}} = -1$  are observed at  $\sim 5\%$  level in both directions for  $z \lesssim 2$  independently of  $n$ . Such behaviour of  $w_{\text{DE}}$  is well admitted by all the most recent observational data, see e.g., Ref. 2). The average value of  $w_{\text{DE}}$  over the interval  $0 \leq z \leq 1$  to which all BAO and most of the SN data refer is very close to  $-1$ . Moreover, in this range (but not for larger values of  $z$ ), the behaviour of  $w_{\text{DE}}$  for minimal allowed values of  $\lambda$  (i.e., for the largest possible deviations from the  $\Lambda\text{CDM}$  background model) is well fitted by the CPL fit<sup>25)</sup>  $w_{\text{DE}}(z) = w_0 + w_a z/(1+z)$  with  $(n, \lambda_{\text{min}}, w_0, w_a) = (2, 0.95, -0.92, -0.23)$ ,  $(3, 0.73, -0.94, -0.22)$ , and  $(4, 0.61, -0.96, -0.21)$ , respectively.  $|1 + w_0|$  and  $|w_a|$  decrease slowly for larger values of  $n$ . These values of  $w_0$  and  $w_a$  lie very close to the center of the 68% and 95% CL ellipses for all combined data in Fig. 13 of Ref. 2).

As explained above, this phantom crossing behaviour is not peculiar to the specific choice of the function (2.2) but a generic one in models that satisfy the stability condition  $\mathcal{F}'' > 0$ . Indeed, a similar behaviour has also been observed in

other  $f(R)$  DE models.<sup>18),26)</sup> We also note that different definitions of  $\rho_{\text{DE}}$ ,  $P_{\text{DE}}$ , and  $w_{\text{DE}}$  have been used in the literature,<sup>27)</sup> which lead to different behaviours of  $w_{\text{DE}}$ .

Although the behaviour of dark energy is quite different depending on model parameters, the total expansion factor  $a_0/a_i$  from the epoch  $\Omega_i = 0.998$  to the present varies only between  $a_0/a_i = 10.8$  and 11, the latter corresponding to the value in the  $\Lambda$ CDM model.

We have also calculated the quantity  $B(z) = (f''/f')(dR/d \ln H)$  (introduced in Ref. 28) at the present time. We have found  $B(0) = 0.21, 6.1 \times 10^{-5}$ , and 0.17, for  $(n, \lambda) = (2, 0.95), (2, 8)$ , and  $(4, 0.61)$ , respectively.

### §3. Density fluctuations

We now turn to the evolution of density fluctuations. In  $f(R)$  gravity, the evolution equation of density fluctuations,  $\delta$ , deeply in the subhorizon regime is given by<sup>29),30)</sup>

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{\text{eff}}\rho\delta = 0, \tag{3.1}$$

where

$$G_{\text{eff}} = \frac{G}{F} \frac{1 + 4\frac{k^2}{a^2} \frac{F'}{F}}{1 + 3\frac{k^2}{a^2} \frac{F'}{F}}, \quad F(R) \equiv f'(R). \tag{3.2}$$

This equation reduces to the correct evolution equation for all wavenumbers for the CDM model in the Einstein gravity where  $F = 1$ .

In the previous paper,<sup>15)</sup> we obtained an analytic solution in the high-curvature regime when the scale factor evolves as  $a(t) \propto t^{2/3}$  and  $F$  takes the asymptotic form

$$F \simeq 1 - 2n\lambda \left(\frac{R}{R_s}\right)^{-2n-1} \equiv 1 - \left(\frac{R}{R_c}\right)^{-N-1}, \tag{3.3}$$

with the following correspondence:

$$N = 2n \quad \text{and} \quad R_c = R_s(2n\lambda)^{1/(2n+1)}. \tag{3.4}$$

The two independent solutions of (3.1) in this regime read

$$\begin{aligned} \delta_{\mathbf{k}}(t) &= \delta_{i\mathbf{k}} \left(\frac{t}{t_i}\right)^{\frac{-1\pm 5}{6}} \\ &\times {}_2F_1\left(\frac{\pm 5 - \sqrt{33}}{4(3N+4)}, \frac{\pm 5 + \sqrt{33}}{4(3N+4)}; 1 \pm \frac{5}{2(3N+4)}; -3\frac{(N+1)k^2}{a_i^2 R_c^2} \left(\frac{t}{t_i}\right)^{2N+8/3}\right) \end{aligned} \tag{3.5}$$

in terms of the hypergeometric function.<sup>15)</sup> In the following discussion, we consider the upper sign solution only, because the other solution corresponds to the decaying mode and is singular at  $t \rightarrow 0$ . Then, the solution behaves as

$$\delta_{\mathbf{k}}(t) \xrightarrow{t \rightarrow 0} \delta_{i\mathbf{k}} \left(\frac{t}{t_i}\right)^{\frac{2}{3}} \quad \text{and} \quad \delta_{\mathbf{k}}(t) \xrightarrow{t \rightarrow \infty} \delta_{i\mathbf{k}} C(k) \left(\frac{t}{t_i}\right)^{\frac{-1+\sqrt{33}}{6}}, \tag{3.6}$$

respectively. The transfer function,  $C(k)$ , is given by

$$\begin{aligned}
 C(k) &= \frac{\Gamma\left(1 + \frac{5}{2(3N+4)}\right) \Gamma\left(\frac{\sqrt{33}}{2(3N+4)}\right)}{\Gamma\left(1 + \frac{5+\sqrt{33}}{4(3N+4)}\right) \Gamma\left(\frac{5+\sqrt{33}}{4(3N+4)}\right)} \left[ \frac{3(N+1)k^2}{a_i^2 R_c} \left(\frac{3R_c t_i^2}{4}\right)^{N+2} \right]^{\frac{-5+\sqrt{33}}{4(3N+4)}} \\
 &= \frac{\Gamma\left(1 + \frac{5}{4(3n+2)}\right) \Gamma\left(\frac{\sqrt{33}}{4(3n+2)}\right)}{\Gamma\left(1 + \frac{5+\sqrt{33}}{8(3n+2)}\right) \Gamma\left(\frac{5+\sqrt{33}}{8(3n+2)}\right)} \left[ \frac{6n\lambda(2n+1)k^2}{a_i^2 R_s} \left(\frac{3R_s t_i^2}{4}\right)^{2(n+1)} \right]^{\frac{-5+\sqrt{33}}{8(3n+2)}} ,
 \end{aligned} \tag{3.7}$$

where

$$t_i = \frac{2}{3} \sqrt{\frac{6}{\lambda R_s}} \sinh^{-1} \sqrt{\frac{1 - \Omega_i}{\Omega_i}} . \tag{3.8}$$

Note that the effective gravitational constant (3.2) reads

$$G_{\text{eff}} = G \left( 1 + \frac{1}{3} \frac{k^2/a^2 m_s^2}{1 + k^2/a^2 m_s^2} \right) \tag{3.9}$$

in the high-curvature regime when  $F \cong 1$ . In the position space, such a theory has the potential

$$V(r) = -\frac{G}{r} \left( 1 + \frac{1}{3} e^{-m_s r} \right) \tag{3.10}$$

per unit mass<sup>33)</sup> for such sufficiently small  $r$  for which the time dependence of  $m_s(t)$  may be neglected. Thus, each Fourier mode feels 4/3 times the conventional gravitational force if and only if  $k/a(t) \gtrsim m_s(t) = (3F')^{-1/2}$ .

The transition from former temporal behaviour to the latter one in (3.6) occurs at the epoch  $t_k$  determined by

$$k = a(t_k) m_s(t_k) = a(t_k) \left( \frac{R_s}{6n(2n+1)\lambda} \right)^{\frac{1}{2}} \left( \frac{R(t_k)}{R_s} \right)^{n+1} . \tag{3.11}$$

The above expression is proportional to  $t_k^{-2n-4/3}$  for those modes whose physical wavenumber (momentum)  $k/a(t)$  crosses the scalaron mass  $m_s(t)$  in the high-curvature regime. This explains the  $k$ -dependence of the transfer function (3.7).<sup>14)</sup> If we adopt an expression of  $R(t)$  in  $\Lambda$ CDM,

$$R(t) = 3H_0^2 \left[ \Omega_{m0} \left( \frac{a_0}{a(t)} \right)^3 + 4(1 - \Omega_{m0}) \right] , \tag{3.12}$$

we can further approximately obtain the crossing time,  $t_*(k)$ , for a smaller wavenumber,  $k_*$ , as well:

$$\frac{k_*}{a(t_*)} = \frac{\lambda^{(n+\frac{1}{2})p_n - \frac{1}{2}}}{\sqrt{6n(2n+1)c_n}^{n+\frac{1}{2}}} \left[ 3\Omega_{m0} \left( \frac{a_0}{a(t_*)} \right)^3 + 12(1 - \Omega_{m0}) \right]^{n+1} H_0 . \tag{3.13}$$

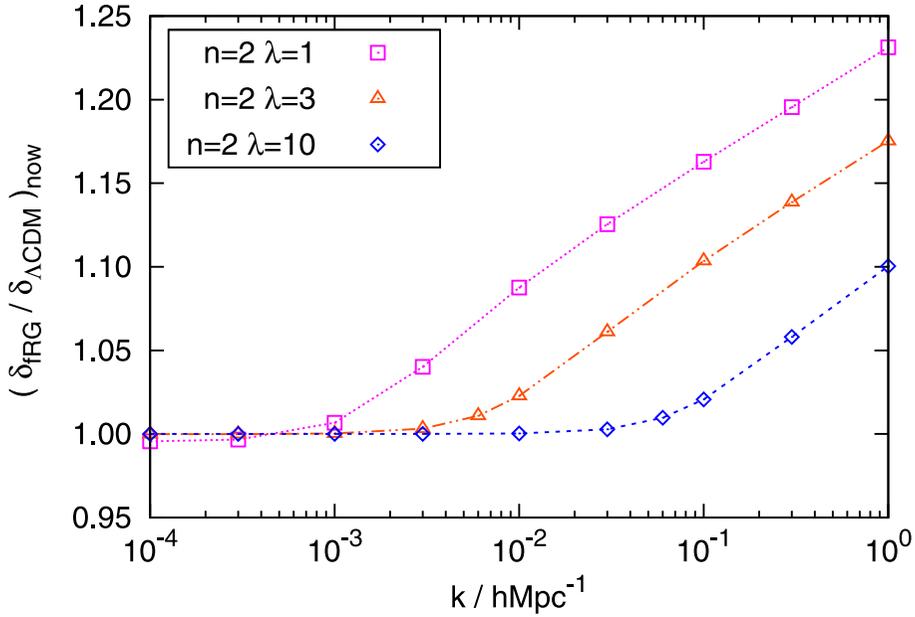


Fig. 2. Ratio of linear density perturbations  $\delta_{\text{IRG}}/\delta_{\Lambda\text{CDM}}$  at present as a function of  $k$  for three different values of  $\lambda$  with  $n = 2$ .

From (3.13), we find that the physical wavenumber crossing the scalaron mass today is given by

$$\frac{k_0}{a_0} = \frac{9.57^{n+1} \lambda^{(n+\frac{1}{2})p_n - \frac{1}{2}}}{\sqrt{6n(2n+1)} c_n^{n+\frac{1}{2}}} H_0 = \begin{cases} 3.2\lambda^{1.88} H_0 & (n = 2) \\ 5.3\lambda^{2.43} H_0 & (n = 3) \\ 5.0\lambda^{2.66} H_0 & (n = 4) \end{cases}. \quad (3.14)$$

Thus, except for cases with large  $\lambda$ , all the observable scales feel the scalaron force today.

Since the analytic solution (3.5) is valid in the high-curvature era only, we must solve (3.1) numerically to obtain a full solution using the analytic solution as an initial condition. Figure 2 shows the ratio of the linear density fluctuation in the  $f(R)$  model,  $\delta_{\text{IRG}}$ , to that in the  $\Lambda\text{CDM}$  model,  $\delta_{\Lambda\text{CDM}}$ , under the same initial condition. Fluctuations with small wavenumbers have practically the same value as those in the  $\Lambda\text{CDM}$  model, while those with larger wavenumbers acquire additional growth due to the scalaron force with the additional power  $k^{\frac{-5+\sqrt{33}}{4(3n+2)}}$  as given in (3.7). From (3.14), the physical wavenumber of this transition is given by

$$\frac{k_0}{a_0} = \begin{cases} 1.07 \times 10^{-3} h \text{ Mpc}^{-1} & (n = 2, \lambda = 1) \\ 8.44 \times 10^{-3} h \text{ Mpc}^{-1} & (n = 2, \lambda = 3), \\ 8.12 \times 10^{-2} h \text{ Mpc}^{-1} & (n = 2, \lambda = 10) \end{cases} \quad (3.15)$$

which explains the figure well.

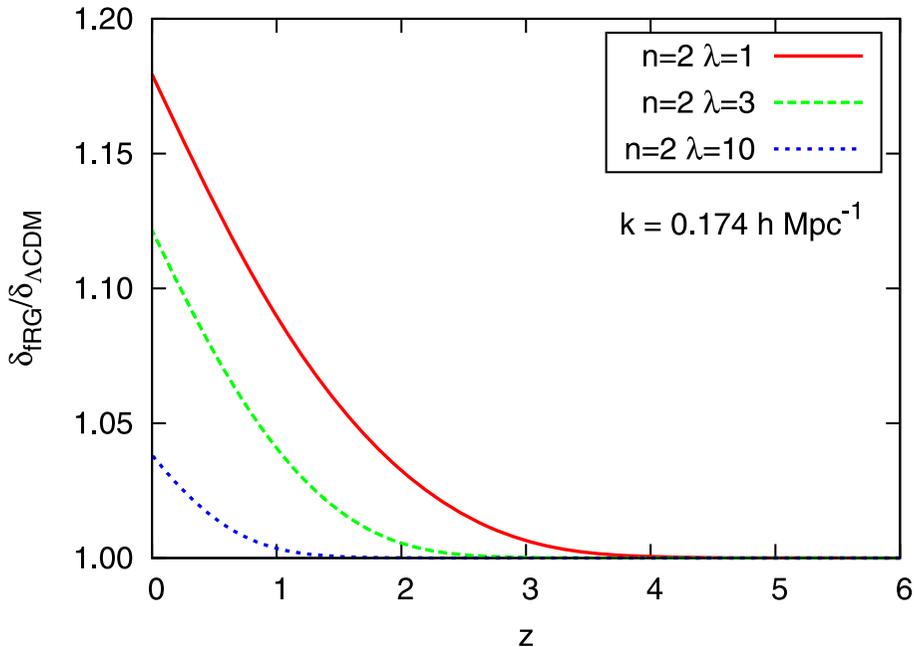


Fig. 3. Ratio of linear density perturbations  $\delta_{\text{FRG}}/\delta_{\Lambda\text{CDM}}(k = 0.174h \text{ Mpc}^{-1})$  as a function of redshift for three different values of  $\lambda$  with  $n = 2$ .

To make a simple comparison of our results with the observations of galaxy clustering, we define an effective wavenumber,  $k_{\text{eff}}(r)$ , corresponding to each length scale  $r$ , in terms of the top-hat mass fluctuation within the same radius:

$$\sigma_r^2 = \int \frac{d^3k}{(2\pi)^3} |W(kr)|^2 P(k) \equiv \frac{4\pi k_{\text{eff}}^3}{(2\pi)^3} P(k_{\text{eff}}(r)), \quad W(kr) \equiv \frac{3j_1(kr)}{kr}. \quad (3.16)$$

Here,  $P(k)$  is the linear matter spectrum obtained using the standard CDM transfer function<sup>31)</sup> with the scale-invariant initial power spectrum of perturbations, i.e., with the primordial spectral index  $n_s = 1$ , and  $W(kr)$  is the Fourier transform of the top-hat window function.

The wavenumber of our particular interest is the scale corresponding to  $\sigma_8$  normalization, for which we find  $k_{\text{eff}}(r = 8h^{-1} \text{ Mpc}) = 0.174h \text{ Mpc}^{-1}$ . Figure 3 shows the redshift evolution of the ratio  $\delta_{\text{FRG}}/\delta_{\Lambda\text{CDM}}$  for this scale for the same values of  $n$  and  $\lambda$  as in Fig. 2. Note that this ratio does not stop growing at the accelerated stage of the Universe expansion, which begins at  $z \approx 0.8$  for  $\lambda = 1$  and  $z \approx 0.75$  for two other values of  $\lambda$ . Since the standard  $\Lambda\text{CDM}$  model normalized by large-scale CMB observations explains galaxy clustering at small scales well,  $\delta_{\text{FRG}}$  should not be much larger than  $\delta_{\Lambda\text{CDM}}$  at these scales. We may typically require  $(\delta_{\text{FRG}}/\delta_{\Lambda\text{CDM}})^2(k = 0.174h \text{ Mpc}^{-1}) \lesssim 1.1$ . Although we neglect nonlinear effects here, the difference between linear calculation and nonlinear N-body simulation remained smaller than 5% at the wavenumber  $0.174h \text{ Mpc}^{-1}$ .<sup>32)</sup>

Figure 4 represents  $(\delta_{\text{FRG}}/\delta_{\Lambda\text{CDM}})^2(k = 0.174h \text{ Mpc}^{-1})$  as a function of  $\lambda$  for  $n = 2, 3$ , and 4. From the analytic formula (3.7), this  $\lambda$  dependence would have the

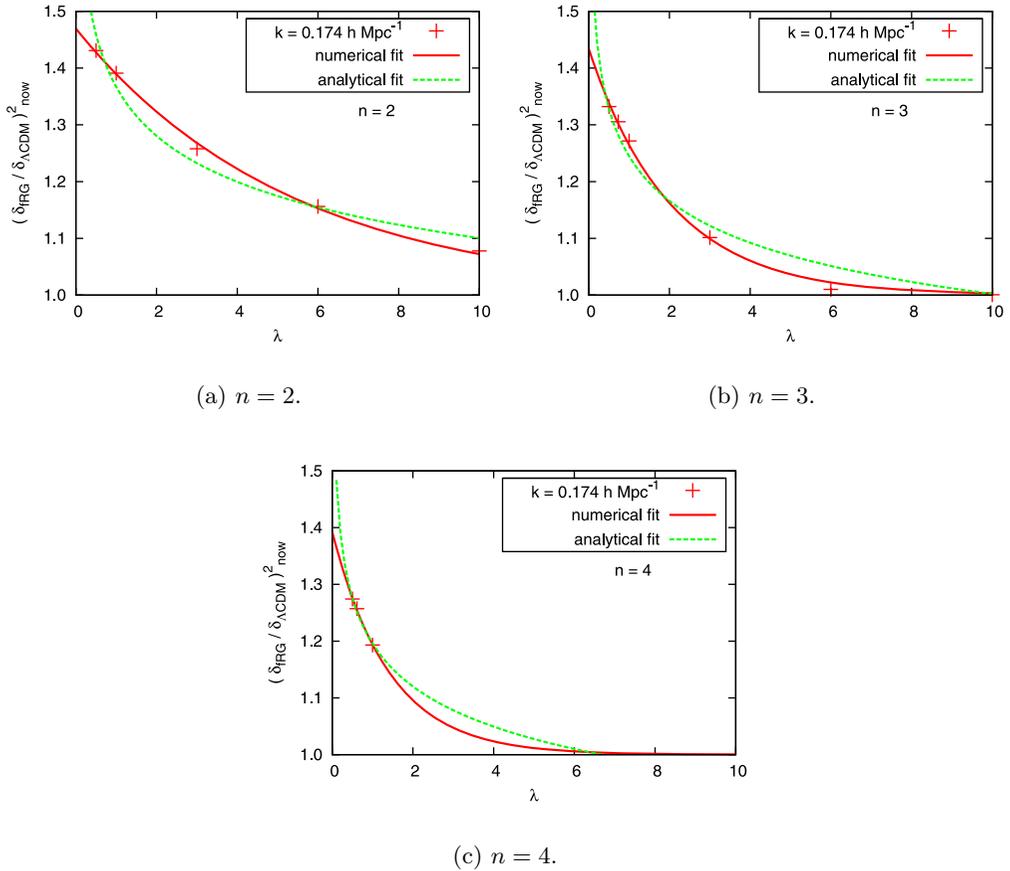


Fig. 4. Present ratio  $(\delta_{\text{fRG}}/\delta_{\Lambda\text{CDM}})^2(k = 0.174h \text{ Mpc}^{-1})$  as a function of  $\lambda$  together with two fitting functions.

form  $(\delta_{\text{fRG}}/\delta_{\Lambda\text{CDM}})^2 \propto C^2(k) \propto \lambda^{-\frac{(2n-p_n+1)(\sqrt{33}-5)}{4(3n+2)}}$ , which is depicted by a broken line in each figure. This curve, however, does not match the asymptotic behaviour  $(\delta_{\text{fRG}}/\delta_{\Lambda\text{CDM}})^2 \rightarrow 1$  for large  $\lambda$ . We find that an exponential function

$$(\delta_{\text{fRG}}/\delta_{\Lambda\text{CDM}})^2 = 1 + b_n e^{-q_n \lambda} \quad (3.17)$$

fits the numerical calculation very well with  $(n, b_n, q_n) = (2, 0.47, 0.19)$ ,  $(3, 0.43, 0.49)$ , and  $(4, 0.39, 0.70)$ , respectively. From these figures, to keep the deviation from the  $\Lambda\text{CDM}$  model smaller than 10% at  $k = 0.174h \text{ Mpc}^{-1}$ , we find that  $\lambda$  should be larger than 8.2, 3.0, and 1.9 for  $n = 2, 3$ , and 4, respectively.

From these analyses, we can constrain the parameter space as shown in Fig. 5. The region that satisfies  $(\delta_{\text{fRG}}/\delta_{\Lambda\text{CDM}})^2(k = 0.174h \text{ Mpc}^{-1}) < 1.1$  corresponds to above the solid line. We also show the 20% boundary with the broken line. The region below the dotted line is forbidden because of the instability of the de Sitter regime.

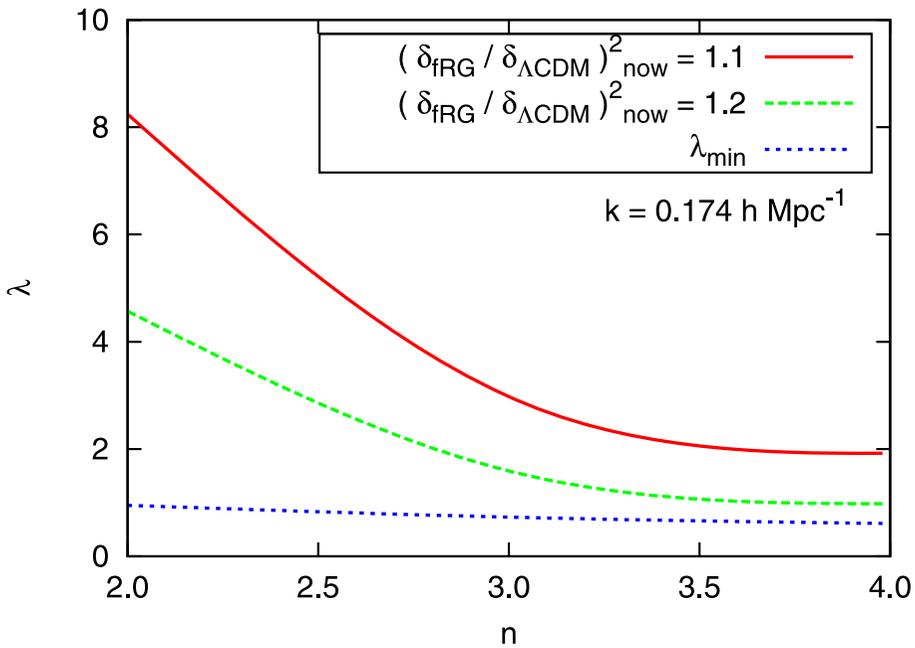


Fig. 5. Constraints for parameter space.

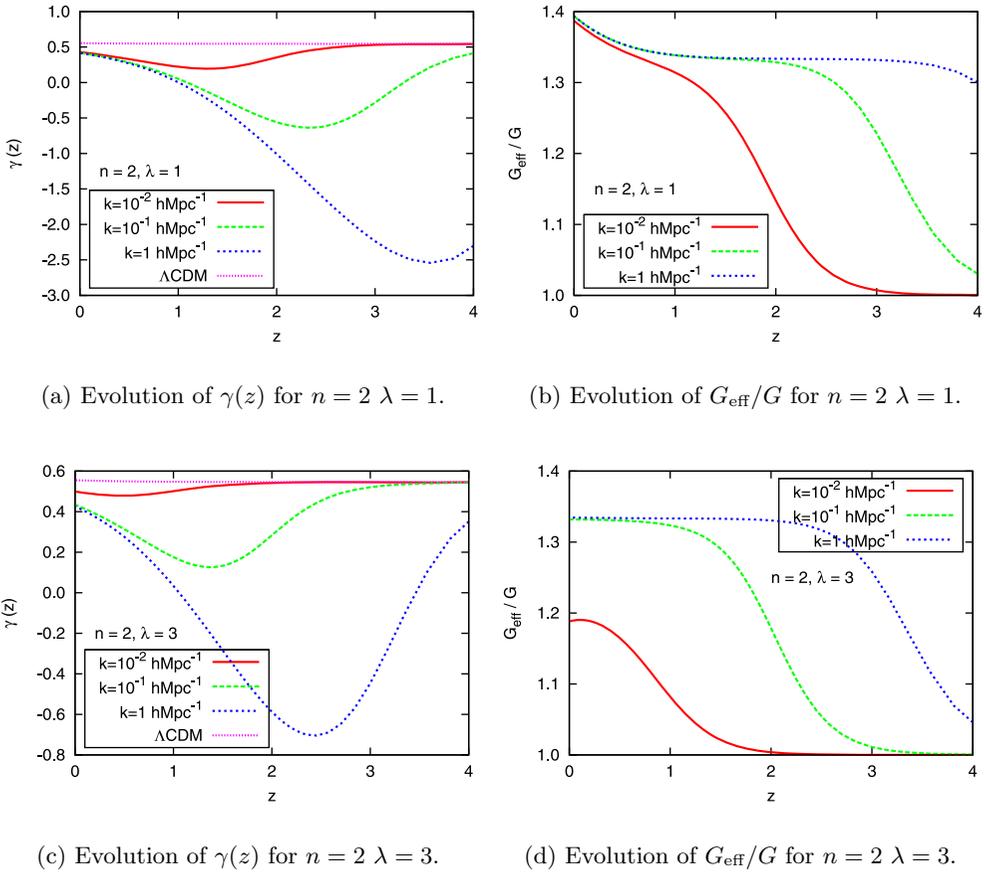
Next, we turn to another important quantity used to distinguish different theories of gravity, namely, the gravitational growth index,  $\gamma(z)$ , of density fluctuations.<sup>33)–38)</sup> It is defined through

$$\frac{d \ln \delta}{d \ln a} = \Omega_m(z)^{\gamma(z)}, \quad \text{or} \quad \gamma(z) = \frac{\log \left( \frac{\dot{\delta}}{H \delta} \right)}{\log \Omega_m}. \quad (3.18)$$

It takes a practically constant value  $\gamma \cong 0.55$  in the standard  $\Lambda$ CDM model,<sup>34)</sup> but it evolves in time in modified gravity theories in general. We also note that  $\gamma(z)$  has a nontrivial  $k$ -dependence in  $f(R)$  gravity, since density fluctuations with different wavenumbers evolve differently. Therefore, this quantity is a useful measure to distinguish modified gravity from the  $\Lambda$ CDM model in the Einstein gravity.

Figure 6 shows the evolution of  $\gamma(z)$  together with that of  $G_{\text{eff}}/G$  for different values of  $k$ . In the early high-redshift regime,  $\gamma(z)$  takes a constant value identical to the  $\Lambda$ CDM model because  $f(R)$  gravity is indistinguishable from the Einstein gravity plus a positive cosmological constant then. It gradually decreases with time, reaches a minimum, and then increases again towards the present epoch. We can understand this tendency from the evolution equation for  $\gamma(z)$ ,<sup>36)</sup>

$$\begin{aligned} & -(1+z) \ln(1 - \Omega_{\text{DE}}) \frac{d\gamma}{dz} \\ & = -(1 - \Omega_{\text{DE}})^\gamma - \frac{1}{2} [1 + 3(2\gamma - 1)w_{\text{DE}}\Omega_{\text{DE}}] + \frac{3}{2} \frac{G_{\text{eff}}}{G} (1 - \Omega_{\text{DE}})^{1-\gamma}, \end{aligned} \quad (3.19)$$

Fig. 6. Evolutions of  $\gamma(z)$  and  $G_{\text{eff}}/G$ .

where  $\Omega_{\text{DE}} = 1 - \Omega_m$  is the density parameter of dark energy based on (2.7). In the high-redshift regime when  $\Omega_{\text{DE}}$  is small, the above equation may be approximated as

$$(1+z)\Omega_{\text{DE}} \frac{d\gamma}{dz} = \frac{3}{2} \left( \frac{G_{\text{eff}}}{G} - 1 \right) + \Omega_{\text{DE}} \left[ \frac{11}{2} \left( \gamma - \frac{6}{11} \right) - \frac{3}{2} (1-\gamma) \left( \frac{G_{\text{eff}}}{G} - 1 \right) - \frac{3}{2} (2\gamma - 1)(w_{\text{DE}} + 1) \right]. \quad (3.20)$$

In the earlier stage, the first term in the right-hand side is more important. That explains why  $\gamma(z)$  starts to decrease when  $G_{\text{eff}}/G$  starts to increase. As time goes by towards lower redshifts, the second term becomes more important to make  $\gamma(z)$  increase again. We note that, recently, Narikawa and Yamamoto<sup>38)</sup> have calculated numerically the time evolution of  $\gamma(z)$  in a simplified model (3.3) and also obtained some analytic expansion, which behaves qualitatively the same as our numerical results but with much more exaggerated amplitudes. Our results, which satisfy all the

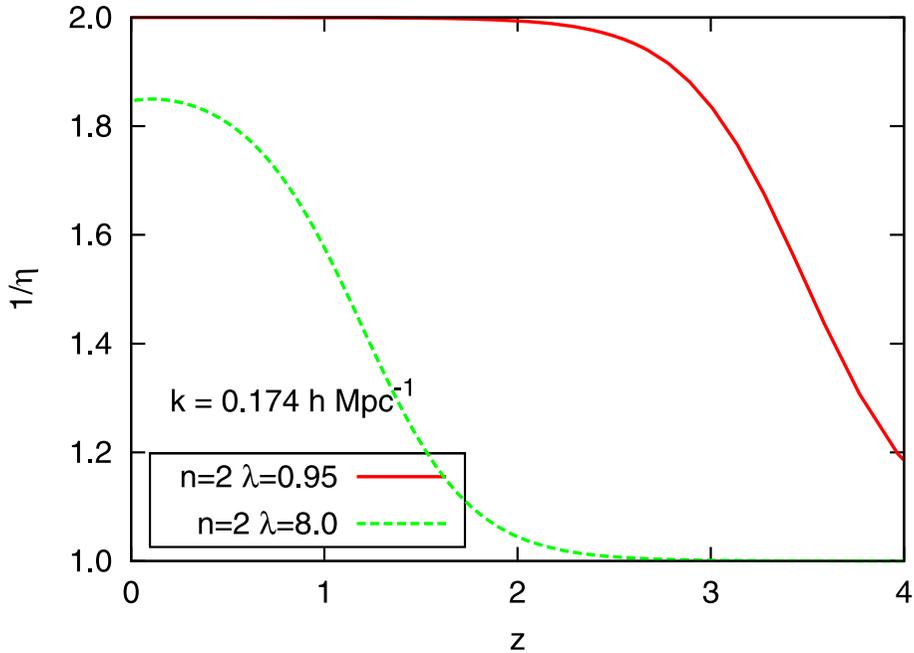


Fig. 7. Evolution of  $1/\eta = \Phi/\Psi$  with  $n = 2$ .

viability conditions, exhibit a milder deviation from the  $\Lambda$ CDM model than those they found. Existing constraints on the growth index<sup>39)</sup> are not strong enough to detect any deviation from the  $\Lambda$ CDM model and/or to obtain new bounds on  $f(R)$  DE models, but future observations may reveal its time and wavenumber dependences.

Another quantity that can characterize the evolution of density perturbations more directly is the ratio  $\delta_{\text{FRG}}(z = 0.5)/\delta_{\text{FRG}}(z = 0)$ . However, it varies only from 0.75 to 0.78 for different choices of the model parameters when the current matter density parameter is fixed to  $\Omega_{m0} = 0.27$  and  $n \geq 2$ . This variation is smaller than that caused by the uncertainty of  $\Omega_{m0}$ .<sup>33)</sup> Thus, at present, it does not help much to single out the best DE model among the considered ones, in contrast to the  $f(R)$  DE model<sup>12)</sup> (it has the same behaviour (3.3) for  $R \gg R_s$ ) in the case corresponding to  $n = 0.5$  in our notations, which has recently been studied by Schmidt et al.<sup>40)</sup>

Finally, we consider the quantity  $1/\eta = \Phi/\Psi$ , namely, the ratio of gravitational potential to curvature perturbation, for which some results from observational data have recently been obtained.<sup>4)</sup> In  $f(R)$  gravity,  $1/\eta$  is expressed as

$$\frac{1}{\eta} = 2 - \frac{1}{1 + 2\frac{k^2 F'}{a^2 F}}. \quad (3.21)$$

Owing to the stability conditions  $F > 0$  and  $F' > 0$ , this quantity always lies between 1 and 2. Thus, stable  $f(R)$  DE models may not explain such a large value of  $1/\eta$ , which is presented in Ref. 4) for the redshift interval  $1 < z < 2$ . Figure 7 shows the evolution of  $1/\eta$  for  $n = 2$  and  $\lambda = 0.95$  (the minimal possible value) and 8.

#### §4. Conclusions

In this paper, we have numerically calculated the evolution of both homogeneous background and density fluctuations in a viable  $f(R)$  DE model based on the specific functional form proposed in Ref. 14). We have found that viable  $f(R)$  gravity models of the present DE and accelerated expansion of the Universe generically exhibit phantom behaviour during the matter-dominated stage with crossing of the phantom boundary  $w_{\text{DE}} = -1$  at redshifts  $z \lesssim 1$ . More exactly, this behaviour is characteristic for all  $f(R)$  DE models that have  $f''(R) > 0$  and a stable future de Sitter epoch and which approach the Einstein gravity sufficiently fast for  $R \gg R_0$ , under the condition that the gravitational constant  $G$  in the Einsteinian representation of the field equations (4) is normalized to its value measured in laboratory experiments (i.e., for  $R \gg R_0$ , too). The predicted time evolution of  $w_{\text{DE}}$  has qualitatively the same behaviour as that recently obtained from observational data.<sup>3)</sup> However, it is important that the condition of stability, or even metastability, of the future de Sitter epoch strongly restricts the possible deviation of  $w_{\text{DE}}$  from  $-1$  by several percents in these models. Thus, the DE phantomness should be small, if it exists at all, which agrees with the present observational data. Still, for the models considered, it is not so hopelessly small as in the case of the similar model<sup>12)</sup> with  $n = 0.5$  recently considered in Ref. 40) using data on cluster abundance. Note also that, in contrast to Ref. 41), we do not impose the so-called thin-shell condition  $|\Delta(f'(R) - 1)| \lesssim |\Phi_N|$ , where  $\Phi_N$  is the Newtonian potential of matter inhomogeneities and  $\Delta$  means change in the quantity in question, for scales exceeding galactic ones where a background matter density approaches the cosmological one. On the other hand, this condition is satisfied automatically for matter overdensities of more than 10 for the parameter range  $n \geq 2$  considered in our paper.

As for the density fluctuations, we have numerically confirmed our previous analytic results of a shift in the power spectrum index for larger wavenumbers that exceed the scalaron mass during the matter-dominated epoch,<sup>15)</sup> while for smaller wavenumbers, fluctuations have the same amplitude as in the  $\Lambda$ CDM model. Once more, the future de Sitter epoch stability condition bounds a possible increase in density fluctuations for cluster scales (compared with the  $\Lambda$ CDM model) by  $\sim 40\%$  for  $n \geq 2$ . On the contrary, if it is proven from observational data that this increase is less than 5%, then the background evolution should be practically indistinguishable from the  $\Lambda$ CDM one:  $|w_{\text{DE}} + 1| < 10^{-4}$  for  $n = 2$ . This shows that  $\sigma_8$  and related density perturbation tests are the most critical ones for the  $f(R)$  DE models considered in the paper. We have obtained that the upper limit on  $|w_{\text{DE}} + 1|$  for  $n = 2$  and  $\lambda = 8$  is  $4.4 \times 10^{-5}$  when  $z = 0.16$ , which is of the same order as  $B(0)$ .

We have also investigated the growth index  $\gamma(k, z)$  of density fluctuations and have presented an explanation of its anomalous evolution in terms of the time dependence of  $G_{\text{eff}}$ . Note that this evolution is characteristic for all  $f(R)$  models in which the scalar particle (scalaron) becomes relativistic ( $k^2 > m_s^2(R)a^2$ ) at recent redshifts. Since  $\gamma$  has characteristic time and wavenumber dependences, future detailed observations may yield useful information on the validity of  $f(R)$  gravity through this quantity, although current constraints have been obtained assuming that it is con-

stant both in time and in wavenumber.<sup>4),39)</sup> Another related observational test of this model is supplied by the large-scale structure of the Universe, which should be different from that in the  $\Lambda$ CDM model. In particular, voids are expected to be more pronounced since the effective gravitational constant is bigger inside them compared with large matter overdensities where it is practically equal to that measured in the laboratory.

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