

Clebsch Potentials in the Variational Principle for a Perfect Fluid

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Equations for a perfect fluid can be obtained by means of the variational principle both in the Lagrangian description and in the Eulerian one. It is known that we need additional fields somehow to describe a rotational isentropic flow in the latter description. We give a simple explanation for these fields; they are introduced to fix both ends of a pathline in the variational calculus. This restriction is imposed in the former description, and should be imposed in the latter description. It is also shown that we can derive a canonical Hamiltonian formulation for a perfect fluid by regarding the velocity field as the input in the framework of control theory.

Subject Index: 518, 519

§1. Introduction

The Euler equation, together with mass and entropy conservations, describes dynamics of the perfect fluid, and can be derived from the variational principle in the Lagrangian description.¹⁾ Let us write ρ for mass per unit *volume*, s for entropy per unit *mass*, and \mathbf{v} for velocity field. Because of the local equilibrium, the internal-energy density per unit *mass*, ϵ , is a function of ρ and s . Apart from constraints coming from the conservation laws, the Lagrangian density is given by

$$\rho \left\{ \frac{1}{2} \mathbf{v}^2 - \epsilon(\rho, s) \right\}. \quad (1.1)$$

The action, which is minimized to yield the Euler equation, is given by the integral of the Lagrangian density over the space and time considered. In the Lagrangian description, the position should be regarded as a variable of the Lagrangian density, and thus the velocity field \mathbf{v} is given by the time derivative of the position of a fluid particle.

In the Eulerian description, not the position but the velocity is a variable of the Lagrangian density. Minimizing the action, we can obtain the Euler equation. However, unlike in the Lagrangian description, the resultant velocity field cannot be rotational on the isentropic condition. To remove this flaw, Bateman²⁾ added some scalar fields, sometimes called Clebsch potentials,^{3),4)} to the Lagrangian density. Later Lin⁵⁾ used more additional scalar fields, which Selinger and Whitham⁶⁾

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considered to be redundant. Schutz⁷⁾ also used Bateman's scalar fields in considering the general-relativistic gravitation field, and still complained of too many additional scalar fields. Irrespective of these controversies, the variational principles^{2),5),7)} in the Eulerian description have been used in some studies on magnetohydrodynamics, multivalued plasmas, elasticity and a hydrodynamic description of relativistic stars.^{6),8),9)} Recently, Kambe¹⁰⁾ has explained Clebsch potentials from the point of view of symmetry of the gauge theory. Yoshida^{11),12)} claimed that the number of required additional fields should be more than Bateman's and fewer than Lin's. Some authors^{1),5),6),10)–15)} considered that Clebsch potentials are related with the Lagrangian coordinates.

We show below that Clebsch potentials in the variational calculus of the Euler description are introduced to fix both ends of a pathline; the same restriction is imposed in that of the Lagrangian description. In relation to the variational principles, various canonical and noncanonical Hamiltonian formulations have been proposed.^{11)–18)} We derive a canonical Hamiltonian formulation by means of an optimal control theory known as Pontryagin's minimum principle.

We state the problem in more detail and show our notation in §2. Our explanation for Clebsch potential is given in §3. We give a brief review of the control theory in §4.1, and derive a canonical formulation by means of the control theory in §4.2. The last section is devoted to discussion. We compare our study with previous works on Clebsch potentials and Hamiltonian formulations in §5. Some details are relegated to Appendices. Although we limit the following discussion in the text to nonrelativistic perfect fluid, we show in Appendix C that our discussion is also valid for a relativistic perfect fluid.

§2. Statement of the problem

We write V for the spacial region of the container filled with a perfect fluid, and ∂V for the surface of the container. We fix the container and consider the dynamics of the fluid from the initial time t_{init} to the final time t_{fin} . Let τ denote time in the Lagrangian description and t denote time in the Eulerian one although they are equivalent in the nonrelativistic theory. The partial derivative with respect to τ (∂_τ), and t (∂_t), implies the Lagrangian and Eulerian time derivatives, respectively. See Appendix A for their relation. We label a fluid particle with its initial position \mathbf{a} , and write $\mathbf{X}(\mathbf{a}, \tau)$ for its position at time τ . Thus, $\mathbf{a} = \mathbf{X}(\mathbf{a}, t_{\text{init}})$ gives the Lagrangian coordinates. The volume element in the Lagrangian coordinates can be given by the determinant of the Jacobian matrix,

$$J(\mathbf{a}, \tau) \equiv \frac{\partial(X_1, X_2, X_3)}{\partial(a_1, a_2, a_3)}, \quad (2.1)$$

where X_i and a_i are, respectively, the components of \mathbf{X} and \mathbf{a} . By definition, we have $J(\mathbf{a}, t_{\text{init}}) = 1$. We assume that a fluid particle never shrinks to a point, i.e., $J(\mathbf{a}, \tau)$ has no singular points, in the space and time considered. Thus, we can define the inverse of $\mathbf{x} = \mathbf{X}(\mathbf{a}, \tau)$, for which we write $\mathbf{a}(\mathbf{x}, t)$. Let us write T for temperature

and p for pressure, and the first law of thermodynamics, $d\epsilon = -pd\rho^{-1} + Tds$, yields

$$p \equiv \rho^2 \left(\frac{\partial \epsilon}{\partial \rho} \right)_s \quad \text{and} \quad T \equiv \left(\frac{\partial \epsilon}{\partial s} \right)_\rho, \tag{2.2}$$

where the subscripts s and ρ indicate variables fixed in the respective partial differentiations. Below, Roman indices run from 1 to 3 except in §4 and Appendix B, and repeated indices are summed up, unless specified otherwise.

2.1. *Lagrangian description*

Replacing \mathbf{v} by $\partial_\tau \mathbf{X}$ in Eq. (1.1), we can make the action in the Lagrangian description. The conservation laws of mass and entropy are given by

$$\rho J = \rho_{\text{init}} \quad \text{and} \quad s = s_{\text{init}}, \tag{2.3}$$

where ρ_{init} and s_{init} denote initial values of ρ and s , respectively. Thus, the action can be defined as

$$S_L[\rho, s, \mathbf{X}, \kappa, \lambda] \equiv \int_{t_{\text{init}}}^{t_{\text{fin}}} d\tau \int_V d^3\mathbf{a} \{ J\mathcal{L}(\rho, s, \partial_\tau \mathbf{X}) + K(\rho J - \rho_{\text{init}}) + \rho\Lambda J(s - s_{\text{init}}) \}, \tag{2.4}$$

where K and Λ are undetermined multipliers introduced to keep the constraints (Eq. (2.3)). In this variational calculus, both ends of a pathline are fixed, i.e.,

$$\delta X_i(\mathbf{a}, t_{\text{init}}) = \delta X_i(\mathbf{a}, t_{\text{fin}}) = 0, \tag{2.5}$$

where δ indicates an infinitesimal variation. Let us write \mathbf{n} for the unit normal vector of ∂V directed outside. The slip boundary condition, $n_i \partial_\tau X_i(\mathbf{a}, \tau) = 0$ if $\mathbf{a} \in \partial V$, means that a fluid particle that is initially in contact with ∂V remains in contact with ∂V although it can slip along the boundary. Thus, we have

$$n_i \delta X_i(\mathbf{a}, \tau) = 0 \quad \text{if} \quad \mathbf{a} \in \partial V. \tag{2.6}$$

The stationary condition of Eq. (2.4) with respect to K , Λ , ρ , s , and X_i is respectively given by the two equations of Eq. (2.3),

$$K = -\frac{1}{2} (\partial_\tau X_i)^2 + h, \tag{2.7}$$

$$\Lambda = T, \tag{2.8}$$

and

$$\rho J \frac{\partial^2}{\partial \tau^2} X_i = -\frac{\partial}{\partial a_j} \left\{ \rho \left(\frac{1}{2} (\partial_\tau X_k)^2 - \epsilon + K \right) \right\} \frac{\partial J}{\partial (\partial X_i / \partial a_j)}, \tag{2.9}$$

where h is enthalpy defined as $h \equiv \epsilon + p/\rho$. The undetermined multipliers K and Λ are related to the physical quantities by means of Eqs. (2.7) and (2.8). Note that surface integral terms, appearing when we applied the integration by parts in calculating Eq. (2.9), vanish because of the boundary conditions (Eq. (2.6)).

From Eqs. (2·7)–(2·9), we successfully obtain the Euler equation in the Lagrangian description,

$$\rho J \frac{\partial^2 X_i}{\partial \tau^2} = - \frac{\partial p}{\partial a_j} \frac{\partial J}{\partial (\partial X_i / \partial a_j)} . \quad (2.10)$$

The above is equivalent to the Euler equation in the Eulerian description, which is given by Eq. (2·21) below. We can show the equivalence by multiplying Eq. (2.10) by $J^{-1}(\mathbf{x}, t)$, and replacing $\partial^2 X_i / \partial \tau^2$ and $J^{-1}(\partial J / \partial (\partial X_i / \partial a_j))$ by $(\partial_t + \mathbf{v} \cdot \nabla)v_i$ and $\partial a_j / \partial x_i$, respectively.¹⁾ As shown in Eq. (2·5), we fix both ends of a pathline. The Euler equation Eq. (2.10) is a second-order differential equation with respect to time τ , and its solution has two constants of integration, which are determined by the fixed ends.

2.2. Eulerian description

In the Eulerian description, ρ , s , and \mathbf{v} are regarded as functions of $\mathbf{x} = (x_1, x_2, x_3)$ and t , and we need not replace v_i by $\partial_\tau X_i$ in Eq. (1·1). The boundary condition Eq. (2·6) gives

$$n_i v_i(\mathbf{x}, t) = 0 \quad \text{if } \mathbf{x} \in \partial V , \quad (2.11)$$

while the conservation laws of mass and entropy Eq. (2·3) can be rewritten into

$$D_t(\rho * 1) = 0 , \quad \text{i.e., } \partial_t \rho = -\nabla \cdot (\rho \mathbf{v}) , \quad (2.12)$$

$$D_t s = 0 , \quad \text{i.e., } \partial_t s = -\mathbf{v} \cdot \nabla s , \quad (2.13)$$

where D_t and $*1$ denote the Lagrangian time derivative equivalent to ∂_τ and volume element equivalent to J , respectively. See Appendix A for the details. If we follow a straightforward way from Eq. (2·4), the action in the Eulerian description should be given by

$$S_E[\rho, s, \mathbf{v}, \kappa, \lambda] \equiv \int_{t_{\text{init}}}^{t_{\text{fin}}} dt \int_V d^3 \mathbf{x} \left\{ \mathcal{L}(\rho, s, \mathbf{v}) - \kappa \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) - \lambda \rho \left(\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s \right) \right\} , \quad (2.14)$$

where κ and λ are undetermined multipliers. Keeping

$$\delta \rho(\mathbf{x}, t_{\text{init}}) = \delta \rho(\mathbf{x}, t_{\text{fin}}) = 0 , \quad (2.15)$$

$$\delta s(\mathbf{x}, t_{\text{init}}) = \delta s(\mathbf{x}, t_{\text{fin}}) = 0 , \quad (2.16)$$

we find the stationary conditions of Eq. (2.14) with respect to κ , λ , \mathbf{v} , ρ , and s to be given respectively by Eqs. (2.12) and (2.13),

$$\mathbf{v} = -\nabla \kappa + \lambda \nabla s , \quad (2.17)$$

$$D_t \kappa = -\frac{1}{2} \mathbf{v}^2 + h , \quad (2.18)$$

and

$$D_t \lambda = T . \quad (2.19)$$

Note that surface integral terms appearing in the calculation vanish because of Eqs. (2.11), (2.15), and (2.16). As discussed later in §4.2, we have the same stationary conditions, Eqs. (2.12), (2.13), and (2.17)–(2.19), even if we replace Eqs. (2.15) and (2.16) by other restrictions of ρ and s at the initial and final times. Comparing Eqs. (2.7) and (2.8) with Eqs (2.18) and (2.19), respectively, we find $K = D_t\kappa$ and $\Lambda = D_t\lambda$. With the aid of Eq. (2.13), the Lagrangian time derivative of Eq. (2.17) yields

$$D_t\mathbf{v} = -\nabla D_t\kappa + (D_t\lambda)\nabla s . \tag{2.20}$$

Substituting Eqs. (2.18) and (2.19) into Eq. (2.20), we obtain the Euler equation

$$\frac{\partial}{\partial t}\mathbf{v} + \frac{1}{2}\nabla v^2 - \mathbf{v} \times (\nabla \times \mathbf{v}) = -\frac{\nabla p}{\rho} . \tag{2.21}$$

From Eq. (2.17), the vorticity is found to be given by

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{v} = \nabla \lambda \times \nabla s . \tag{2.22}$$

Although the derivation of the Euler equation appears successful, the resultant vorticity Eq. (2.22) vanishes on the isentropic condition $\nabla s = 0$. It is the flaw mentioned in the introduction.

§3. Fixing ends of a pathline

Both ends of a pathline are fixed in the Lagrangian variational calculus, as shown in Eq. (2.5). It is thus natural that the ends are fixed in the Eulerian variational calculus, but not in §2.2.

Let A_i ($i = 1, 2, 3$) be three scalar fields so that a pathline coincides with an intersection of hypersurfaces given by

$$A_i(\mathbf{x}, t) = \text{constant for } i = 1, 2, 3 . \tag{3.1}$$

The determinant of the Jacobian matrix

$$\frac{\partial(A_1, A_2, A_3)}{\partial(x_1, x_2, x_3)} = (\nabla A_1 \times \nabla A_2) \cdot \nabla A_3 \tag{3.2}$$

gives a reciprocal of the volume element in the coordinates defined in terms of \mathbf{A} . The mass conservation, the first equation of Eq. (2.3), is also represented by

$$\rho(\mathbf{a}(\mathbf{x}, t), t_{\text{init}}) (\nabla A_1 \times \nabla A_2) \cdot \nabla A_3 = \rho(\mathbf{x}, t) , \tag{3.3}$$

where \mathbf{A} has variables \mathbf{x} and t .

We can take \mathbf{A} for the Lagrangian coordinate \mathbf{a} . Otherwise, we can consider \mathbf{A} as an invertible function of \mathbf{a} . The indefiniteness was also pointed out in Ref. 10). Suppose, for example, we have

$$A_1 = a_1 , A_2 = a_2 , \text{ and } A_3 = a_3 + f(a_1, a_2) , \tag{3.4}$$

where $f(a_1, a_2)$ is a two-variable function of a_1 and a_2 . Because we have

$$\nabla A_3 = \nabla a_3 + \frac{\partial f}{\partial a_1} \nabla a_1 + \frac{\partial f}{\partial a_2} \nabla a_2 , \quad (3.5)$$

we can define ∇A_3 so that it is normal to $\nabla A_1 (= \nabla a_1)$ and $\nabla A_2 (= \nabla a_2)$ by tuning the function f . If we take $\tilde{\mathbf{A}}$ to be an arbitrary invertible function of \mathbf{a} , and substitute $\tilde{\mathbf{A}}$, instead of \mathbf{a} , into Eqs. (3.4) and (3.5), we can also make ∇A_3 normal to $\nabla A_1 (= \nabla \tilde{A}_1)$ and $\nabla A_2 (= \nabla \tilde{A}_2)$. By using it, we can rewrite Eq. (3.3) into

$$\nabla A_3 = \frac{\rho(\mathbf{x}, t)}{\rho(\mathbf{a}(\mathbf{x}, t), t_{\text{init}})(\nabla A_1 \times \nabla A_2)} , \quad (3.6)$$

where A_i has variables \mathbf{x} and t .

Let us impose

$$\delta A_\alpha(\mathbf{x}, t_{\text{init}}) = \delta A_\alpha(\mathbf{x}, t_{\text{fin}}) = 0, \text{ for } \alpha = 1, 2 . \quad (3.7)$$

Using Eqs. (2.15), (3.6), and (3.7), we have

$$\delta A_3(\mathbf{x}, t_{\text{init}}) = \delta A_3(\mathbf{x}, t_{\text{fin}}) = 0 . \quad (3.8)$$

Thus, we can fix the ends of a pathline in the variational calculus by imposing Eqs. (2.15) and (3.7). Note that the values of A_1 and A_2 can be determined independent of the mass density ρ .

Since \mathbf{v} is tangent to the hypersurfaces, A_1 and A_2 must satisfy

$$D_t A_\alpha = 0 , \text{ i.e., } \frac{\partial}{\partial t} A_\alpha = -\mathbf{v} \cdot \nabla A_\alpha . \quad (3.9)$$

Hence, in the Eulerian description, we should minimize the action

$$S_I[\rho, s, \mathbf{v}, \mathbf{A}, \kappa, \lambda, \boldsymbol{\beta}] \equiv S_E[\rho, s, \mathbf{v}, \kappa, \lambda] - \int_{t_{\text{init}}}^{t_{\text{fin}}} dt \int_V dx^3 \sum_{\alpha=1}^2 \rho \beta_\alpha D_t A_\alpha , \quad (3.10)$$

with Eqs. (2.15), (2.16), and (3.7) kept. Here, β_α is the undetermined multiplier, and we write \mathbf{A} and $\boldsymbol{\beta}$ for (A_1, A_2) and (β_1, β_2) respectively. The stationary conditions of Eq. (3.10) with respect to A_α are given by

$$D_t \beta_\alpha = 0 , \text{ i.e., } \frac{\partial}{\partial t} \beta_\alpha = -\mathbf{v} \cdot \nabla \beta_\alpha . \quad (3.11)$$

Here, surface integration terms appearing in the integration by parts vanish because of Eqs. (2.11) and (3.7). The stationary conditions with respect to \mathbf{v} and ρ are respectively given by

$$\mathbf{v} = -\nabla \kappa + \lambda \nabla s + \sum_{\alpha=1}^2 \beta_\alpha \nabla A_\alpha , \quad (3.12)$$

and

$$D_t \kappa = -\frac{1}{2} \mathbf{v}^2 + h + \sum_{\alpha=1}^2 \beta_\alpha D_t A_\alpha , \quad (3.13)$$

which equals Eq. (2·18) because of Eq. (3·9). The other stationary conditions with respect to κ , λ , ρ , and s are respectively given by Eqs. (2·12), (2·13), (2·18), and (2·19). We obtain the Euler equation Eq. (2·21) from Eq. (3·12) in the same way as we used in the preceding section. Values of β_1 and β_2 are determined by the fixed ends, as discussed later in §4.2. From Eq. (3·12), the vorticity is found to be given by

$$\boldsymbol{\omega} = \nabla\lambda \times \nabla s + \sum_{\alpha=1}^2 \nabla\beta_\alpha \times \nabla A_\alpha . \tag{3·14}$$

This term $\sum_{\alpha=1}^2 \nabla\beta_\alpha \times \nabla A_\alpha$ makes the flow rotational even on the isentropic condition $\nabla s = 0$, and the flaw mentioned in the introduction is removed.

§4. Hamiltonian formulation

Hamiltonian formulations in analytical mechanics can be regarded as a special case of more generalized formulation in control theory, known as Pontryagin’s minimum principle.^{19),20)} In §4.1, we first give a brief review of this theory. In §4.2, we apply it to the dynamics of a perfect fluid to derive one of the canonical Hamiltonian formulations with Clebsch potentials, which was previously derived in a different way.^{12),18)}

4.1. Brief review of the control theory

Let \mathbf{q} represent the state of a system to be controlled, and \mathbf{u} represent the input to this system, and we assume the time evolution of the state to be given in terms of a function of the state and input as

$$\frac{d}{dt}\mathbf{q} = \mathbf{F}(\mathbf{q}, \mathbf{u}) . \tag{4·1}$$

The optimal input is determined so that a cost functional

$$\int_{t_{\text{init}}}^{t_{\text{fin}}} dt L(\mathbf{q}(t), \mathbf{u}(t)) , \tag{4·2}$$

where L denotes a function, is minimized on condition that the initial and final states are fixed, i.e.,

$$\delta\mathbf{q}(t_{\text{init}}) = 0 , \tag{4·3}$$

and

$$\delta\mathbf{q}(t_{\text{fin}}) = 0 . \tag{4·4}$$

We define the undetermined multiplier, \mathbf{p} , which is also called costate. The optimal input is obtained by minimizing

$$\begin{aligned} S[\mathbf{q}, \mathbf{p}, \mathbf{u}] &= \int_{t_{\text{init}}}^{t_{\text{fin}}} dt \left\{ L(\mathbf{q}, \mathbf{u}) + \mathbf{p} \cdot \left(\frac{d}{dt}\mathbf{q} - \mathbf{F}(\mathbf{q}, \mathbf{u}) \right) \right\} \\ &= \int_{t_{\text{init}}}^{t_{\text{fin}}} dt \left\{ -H(\mathbf{q}, \mathbf{p}, \mathbf{u}) + \mathbf{p} \cdot \frac{d}{dt}\mathbf{q} \right\} , \end{aligned} \tag{4·5}$$

where $H(\mathbf{q}, \mathbf{p}, \mathbf{u})$ is defined as

$$H(\mathbf{q}, \mathbf{p}, \mathbf{u}) \equiv -L(\mathbf{q}, \mathbf{u}) + \mathbf{p} \cdot \mathbf{F}(\mathbf{q}, \mathbf{u}) . \quad (4.6)$$

Let $\mathbf{u}^*(\mathbf{q}, \mathbf{p})$ denote the input minimizing Eq. (4.5) on condition that \mathbf{q} and \mathbf{p} are given, and it is necessary for \mathbf{u}^* to satisfy

$$\frac{\partial H(\mathbf{q}, \mathbf{p}, \mathbf{u}^*)}{\partial u_i^*} = 0 . \quad (4.7)$$

Introducing

$$H^*(\mathbf{q}, \mathbf{p}) \equiv H(\mathbf{q}, \mathbf{p}, \mathbf{u}^*(\mathbf{q}, \mathbf{p})) , \quad (4.8)$$

we define the preoptimized action as

$$S^*[\mathbf{q}, \mathbf{p}] \equiv \int_{t_{\text{init}}}^{t_{\text{fin}}} dt \left\{ -H^*(\mathbf{q}, \mathbf{p}) + \mathbf{p} \cdot \frac{d}{dt} \mathbf{q} \right\} , \quad (4.9)$$

which is not larger than $S[\mathbf{q}, \mathbf{p}, \mathbf{u}]$. We can obtain the optimal input by solving the stationary conditions of Eq. (4.9) with respect to \mathbf{p} and \mathbf{q} , which are given respectively by

$$\frac{dq_i}{dt} = \frac{\partial H^*(\mathbf{q}, \mathbf{p})}{\partial p_i} \quad (4.10)$$

and

$$\frac{dp_i}{dt} = -\frac{\partial H^*(\mathbf{q}, \mathbf{p})}{\partial q_i} . \quad (4.11)$$

The boundary conditions for these equations are given by the initial and final states, which are assumed to be fixed in Eqs. (4.3) and (4.4). If we impose not Eq. (4.4) but a boundary condition

$$\mathbf{p}(t_{\text{fin}}) = 0 , \quad (4.12)$$

on the stationary conditions of Eq. (4.9), we can derive Eqs. (4.10) and (4.11). As discussed in Ref. 20), we can have more general conditions for the initial and final states. These conditions are called the transversality conditions in control theory.

Let us assume \mathbf{q} to represent the position of a material particle, and define \mathbf{F} as

$$\mathbf{F}(\mathbf{q}, \mathbf{u}) \equiv \mathbf{u} , \quad (4.13)$$

and we find the formulation above to be equivalent to that of analytical mechanics for the particle.

4.2. An application for a perfect fluid

We can generalize the formulation in the preceding subsection to cases where the state variable and input variable are functions of space and time, respectively. Let us consider ρ, s , and \mathbf{A} in §3 to represent the state, and \mathbf{v} to represent the input variable. We can identify Eq. (4.1) with a set of Eqs. (2.12), (2.13), and (3.9) by defining $\mathbf{q} \equiv (\rho, s, \mathbf{A})$ and $\mathbf{u} \equiv \mathbf{v}$. The costate is defined as $\mathbf{p} \equiv (-\kappa, -\rho\lambda, -\rho\boldsymbol{\beta})$.

The variations at the initial and final times satisfy Eqs. (2·15), (2·16), and (3·7). The function corresponding to Eq. (4·6) is given by

$$\mathcal{H}(\mathbf{q}, \mathbf{p}, \mathbf{u}) \equiv -\mathcal{L} + \kappa \nabla \cdot (\rho \mathbf{v}) + \rho \lambda \mathbf{v} \cdot \nabla s + \sum_{\alpha=1}^2 \rho \beta_{\alpha} \nabla A_{\alpha} . \quad (4.14)$$

Using Eqs. (1·1) and (2·11), we find

$$\int_V d^3 \mathbf{x} \mathcal{H}(\mathbf{q}, \mathbf{p}, \mathbf{u}) = \int_V d^3 \mathbf{x} \left\{ \mathcal{H}^*(\mathbf{q}, \mathbf{p}) - \frac{\rho}{2} \left(\mathbf{v} + \nabla \kappa - \lambda \nabla s - \sum_{\alpha=1}^2 \beta_{\alpha} \nabla A_{\alpha} \right)^2 \right\} , \quad (4.15)$$

where we define

$$\mathcal{H}^*(\mathbf{q}, \mathbf{p}) \equiv \rho \left\{ \epsilon(\rho, s) + \frac{1}{2} \left(-\nabla \kappa + \lambda \nabla s + \sum_{\alpha=1}^2 \beta_{\alpha} \nabla A_{\alpha} \right)^2 \right\} . \quad (4.16)$$

Thus, the velocity field given by Eq. (3·12) turns out to minimize the action on condition that the state and costate are given. We find that Eq. (3·10) satisfies

$$\begin{aligned} S_l[\mathbf{q}, \mathbf{p}, \mathbf{u}] &= \int_{t_{\text{init}}}^{t_{\text{fin}}} dt \int_V d^3 \mathbf{x} \left\{ -\mathcal{H}(\mathbf{q}, \mathbf{p}, \mathbf{u}) + \mathbf{p} \cdot \frac{d\mathbf{q}}{dt} \right\} \\ &\geq \int_{t_{\text{init}}}^{t_{\text{fin}}} dt \int_V d^3 \mathbf{x} \left\{ -\mathcal{H}^*(\mathbf{q}, \mathbf{p}) + \mathbf{p} \cdot \frac{d\mathbf{q}}{dt} \right\} , \end{aligned} \quad (4.17)$$

which is the preoptimized action, $S_l^*[\mathbf{q}, \mathbf{p}]$. Here, we have

$$\mathbf{p} \cdot \frac{d\mathbf{q}}{dt} = -\kappa \frac{\partial \rho}{\partial t} - \rho \lambda \frac{\partial s}{\partial t} - \sum_{\alpha=1}^2 \rho \beta_{\alpha} \frac{\partial A_{\alpha}}{\partial t} . \quad (4.18)$$

The stationary condition of the preoptimized action gives a set of Eqs. (2·12), (2·13), and (3·9), and a set of Eqs. (2·18), (2·19), and (3·11). These sets can be rewritten respectively into

$$\frac{\partial q_i(\mathbf{x}, t)}{\partial t} = \frac{\partial \mathcal{H}^*(\mathbf{q}, \mathbf{p})}{\partial p_i} - \sum_{j=1}^3 \frac{\partial}{\partial x_j} \frac{\partial \mathcal{H}^*(\mathbf{q}, \mathbf{p})}{\partial (\partial p_i / \partial x_j)} \quad (4.19)$$

and

$$\frac{\partial p_i(\mathbf{x}, t)}{\partial t} = -\frac{\partial \mathcal{H}^*(\mathbf{q}, \mathbf{p})}{\partial q_i} + \sum_{j=1}^3 \frac{\partial}{\partial x_j} \frac{\partial \mathcal{H}^*(\mathbf{q}, \mathbf{p})}{\partial (\partial q_i / \partial x_j)} , \quad (4.20)$$

for $i = 1, 2, 3, 4$, which correspond to Eqs. (4·10) and (4·11), respectively. We can rewrite Eqs. (4·19) and (4·20) respectively into

$$q_i(\mathbf{x}, t_1) = q_i(\mathbf{x}, t_{\text{init}}) + \int_{t_{\text{init}}}^{t_1} dt \left\{ \frac{\partial \mathcal{H}^*(\mathbf{q}, \mathbf{p})}{\partial p_i} - \sum_{j=1}^3 \frac{\partial}{\partial x_j} \frac{\partial \mathcal{H}^*(\mathbf{q}, \mathbf{p})}{\partial (\partial p_i / \partial x_j)} \right\} , \quad (4.21)$$

$$p_i(\mathbf{x}, t_1) = p_i(\mathbf{x}, t_{\text{init}}) + \int_{t_{\text{init}}}^{t_1} dt \left\{ -\frac{\partial \mathcal{H}^*(\mathbf{q}, \mathbf{p})}{\partial q_i} + \sum_{j=1}^3 \frac{\partial}{\partial x_j} \frac{\partial \mathcal{H}^*(\mathbf{q}, \mathbf{p})}{\partial (\partial q_i / \partial x_j)} \right\} . \quad (4.22)$$

In §3, we fixed the initial and final states by imposing Eqs. (2·15), (2·16), and (3·7). These restrictions can be rewritten into Eqs. (4·3) and (4·4). The initial value of the costate $\mathbf{p}(\mathbf{x}, t_{\text{init}})$, i.e., $\kappa(\mathbf{x}, t_{\text{init}})$, $\lambda(\mathbf{x}, t_{\text{init}})$, and $\boldsymbol{\beta}(\mathbf{x}, t_{\text{init}})$, should be determined so that the given values of $\mathbf{q}(\mathbf{x}, t_{\text{init}})$ and $\mathbf{q}(\mathbf{x}, t_{\text{fin}})$ are compatible with Eqs. (4·21) and (4·22).

We can relax the restrictions in the variational calculus. First, suppose that we do not impose Eq. (3·7) for the final time. As discussed in relation to Eq. (4·12), we have $\boldsymbol{\beta}(\mathbf{x}, t_{\text{fin}}) = \mathbf{0}$. By substituting it into Eq. (3·11), we obtain

$$\boldsymbol{\beta}(\mathbf{x}, t) = \mathbf{0} . \quad (4\cdot23)$$

Thus, the velocity field Eq. (3·12) becomes irrotational on the isentropic condition, like the velocity given by Eq. (2·17). Next, suppose we do not impose any of Eqs. (2·15), (2·16), and (3·7) for the final time, and we have Eq. (4·23) and

$$\kappa(\mathbf{x}, t_{\text{fin}}) = \lambda(\mathbf{x}, t_{\text{fin}}) = 0 . \quad (4\cdot24)$$

If we can solve Eqs. (4·21) and (4·22) with their boundary conditions above, the solution uniquely determines $\rho, s, \mathbf{A}, \kappa, \lambda$, and $\boldsymbol{\beta}$, and thus \mathbf{v} , for any \mathbf{x} and t considered.

§5. Discussion

Various actions for a perfect fluid in the Eulerian description were proposed in previous works so that a rotational isentropic flow can be derived. Bateman²⁾ proposed an action with additional scalar field ζ and η as

$$S_b[\rho, s, \mathbf{v}, \kappa, \lambda, \eta, \zeta] \equiv S_E[\rho, s, \mathbf{v}, \kappa, \lambda] - \int dt dx^3 \rho \zeta D_t \eta , \quad (5\cdot1)$$

and derived

$$\mathbf{v} = -\nabla\kappa + \lambda\nabla s + \zeta\nabla\eta . \quad (5\cdot2)$$

On the isentropic condition, the velocity field is given by

$$\mathbf{v} = -\nabla\kappa + \zeta\nabla\eta . \quad (5\cdot3)$$

The additional field ζ and η are sometimes called Clebsch potentials because, apart from the variational principle, Clebsch^{3),4)} had shown that a velocity field of a perfect fluid is given by Eq. (5·3), provided that the vorticity can be put in the form

$$\boldsymbol{\omega} = \nabla\zeta \times \nabla\eta . \quad (5\cdot4)$$

Lin⁵⁾ introduced more scalar fields to identify a fluid particle, mentioned the conservation law of identity without sufficient explanation of physical meanings of the identities and their conservation, and derived

$$\mathbf{v} = -\nabla\kappa + \lambda\nabla s + \sum_{i=1}^3 \beta_i \nabla A_i , \quad (5\cdot5)$$

which can be rotational on the isentropic condition. He considered A_i to be a function of the original position of the fluid particle. Later, Selinger and Whitham⁶⁾ mentioned, “Lin’s device still remains somewhat mysterious from a strictly mathematical viewpoint, but necessary for it seems to be firmly established as we proceed,” and also commented, “But it was noted by Clebsch that any flow velocity can be represented by” Eq. (5·3). This statement would be misleading because Eq. (5·3) is valid if and only if Eq. (5·4) holds.⁴⁾ Schutz⁷⁾ generalized Bateman’s variational principle for a relativistic perfect fluid, and mentioned, “The existence of an Eulerian variational principle may be a beginning. What is needed, I believe, is a Hamiltonian principle in a minimum number of variables. The present action principle seems to have ‘too many’ free variables”. Yoshida¹²⁾ pointed out that an arbitrary velocity field can be given by Eq. (3·12), not Eq. (5·3), in a way different from the way we used in §3. There, we first introduced A_i ($i = 1, 2, 3$), which can be the Lagrangian coordinates, but can be otherwise. Using the mass conservation Eq. (3·6), we next showed that we can fix both ends of a pathline without using A_3 .

Clebsch potentials have been considered to be related with the Lagrangian coordinates.^{1), 5), 6), 10), 11), 13), 15)} Selinger and Whitman⁶⁾ mentioned that Clebsch potentials satisfy Eq. (3·9). Kambe¹⁰⁾ considered Eq. (5·5) to give the transformation between the Lagrangian and Eulerian coordinates, and mentioned the indefiniteness of Clebsch potentials. Yoshida¹¹⁾ claimed that Clebsch potentials provide a one-to-one map between the Lagrangian densities in the Lagrangian and Eulerian descriptions.

Although Clebsch potentials have been considered to label a fluid particle, why the label is required in the Eulerian variational calculus has not been clear in previous studies. We can fix both ends of a path line in the Eulerian variational calculus by imposing Eq. (3·7) on the Clebsch potentials. In doing so, we can keep the same conditions in the variational calculus, as in the Lagrangian description. Thus, we can start with the same Lagrangian density to obtain the same velocity field, which can be irrotational on the isentropic condition, as in the Lagrangian description. As discussed in §4.2, if we do not impose Eq. (3·7), i.e., do not fix both ends of a path line, the resultant Eulerian variational calculus has no corresponding Lagrangian variational calculus, in which both ends of a path line are always fixed by Eq. (2·5).

As referred to in Ref. 10), Lin’s variational principle yields

$$D_t \beta_i = 0, \quad \text{for } i = 1, 2, 3 \tag{5·6}$$

which means that β_i is a function of the Lagrangian coordinates, i.e., Clebsch potentials A_i ($i = 1, 2, 3$), and the vorticity on the isentropic condition is given by

$$\omega(A_1, A_2, A_3) = \sum_{i,j=1}^3 \frac{\partial \beta_i(\mathbf{A})}{\partial A_j} \nabla A_j \times \nabla A_i . \tag{5·7}$$

We showed that A_3 is a function of ρ , A_1 , and A_2 in Eq. (3·6). Thus, the values of β_1 and β_2 are given by functions of ρ , A_1 , and A_2 because of Eq. (3·11). Hence, the

vorticity on the isentropic condition can be rewritten into

$$\omega(A_1, A_2, \rho) = \left(\frac{\partial \beta_2}{\partial A_1} - \frac{\partial \beta_1}{\partial A_2} \right) \nabla A_1 \times \nabla A_2 + \sum_{\alpha=1}^2 \frac{\partial \beta_\alpha}{\partial \rho} \nabla \rho \times \nabla A_\alpha . \quad (5.8)$$

Hamiltonian formulations for a perfect fluid also have been studied by some authors.^{11)–18)} Arnold and Khesin¹⁶⁾ studied a Hamiltonian formulation of an incompressible isentropic flow, and Morrison and Greene¹⁷⁾ discovered an extremely complicated Eulerian Poisson bracket for an isentropic perfect fluid. Holm and Kupershmidt¹³⁾ modified the Hamiltonian formulations^{16), 17)} by means of Clebsch potentials. As pointed out in Ref. 11), Casimir invariants, such as helicity, constrain the dynamics of a perfect fluid in these noncanonical Hamiltonian formulations.^{13)–17)} Note that the helicity is conserved only for a barotropic perfect fluid and an isentropic perfect fluid. On the other hand, as discussed in Appendix B, there is no Casimir invariant in a canonical Hamiltonian formulation. We obtain the same canonical Hamiltonian formulation as derived in Ref. 12) by regarding the velocity as the input variable.

We have assumed fluid dynamics in a fixed container. This assumption can be relaxed; our discussion remains valid when we consider the dynamics of a perfect fluid in an infinitely large spacial region and assume that the mass density tends to become zero far from a certain point. Our discussion can be applied to a relativistic perfect fluid, where we should start with the Lagrangian density slightly different from Eq. (1.1). We show the details in Appendix C for completeness of our study.

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Appendix A

— Lagrangian Time Derivative —

The Lagrangian time derivative $\partial_\tau \equiv D_t$ is given by $\partial_t + \mathbf{v} \cdot \nabla$ for a scalar, $\partial_t + \nabla(\mathbf{v} \cdot) - \mathbf{v} \times \nabla \times$ for a cotangent vector, $\partial_t - \nabla \times (\mathbf{v} \times) + \mathbf{v}(\nabla \cdot)$ for an axial vector, and $\partial_t + \nabla \cdot (\mathbf{v})$ for the volume. These expressions are unified as

$$D_t \equiv \partial_t + L_{\mathbf{v}} , \quad (\text{A} \cdot 1)$$

where $L_{\mathbf{v}}$ denotes the Lie derivative along the vector field \mathbf{v} . Note that D_t is commutative with exterior derivative d , and thus gradient $\nabla \cdot$ and rotation $\nabla \times$. Because ρ and s are 0-forms, and the volume element $*1$ is a 3-form,²¹⁾ the conservation laws of mass and entropy are respectively found to be given by

$$D_t(\rho * 1) = 0 \quad \text{and} \quad D_t s = 0 . \quad (\text{A} \cdot 2)$$

Appendix B

— Canonical Poisson Bracket —

Let f and g denote functions of \mathbf{q} and \mathbf{p} , which are N -dimensional vectors, and we define their Poisson bracket as

$$\{f, g\} = \sum_{i=1}^N \left[\frac{\delta f}{\delta q_i} \frac{\delta g}{\delta p_i} - \frac{\delta f}{\delta p_i} \frac{\delta g}{\delta q_i} \right], \tag{B.1}$$

where we define

$$\frac{\delta}{\delta q_i} \equiv \frac{\partial}{\partial q_i} - \sum_{j=1}^3 \frac{\partial}{\partial x_j} \frac{\partial}{\partial (\partial q_i / \partial x_j)}, \tag{B.2}$$

$$\frac{\delta}{\delta p_i} \equiv \frac{\partial}{\partial p_i} - \sum_{j=1}^3 \frac{\partial}{\partial x_j} \frac{\partial}{\partial (\partial p_i / \partial x_j)}. \tag{B.3}$$

We can rewrite Eqs. (4.10) and (4.11) into

$$\frac{d}{dt} \mathbf{q} = \{\mathbf{q}, \mathcal{H}^*(\mathbf{q}, \mathbf{p})\}, \tag{B.4}$$

$$\frac{d}{dt} \mathbf{p} = \{\mathbf{p}, \mathcal{H}^*(\mathbf{q}, \mathbf{p})\}, \tag{B.5}$$

respectively. This Hamiltonian formulation is canonical because the associated Poisson bracket can be rewritten by means of the symplectic matrix, i.e.,

$$\{f, g\} = \left(\frac{\delta f}{\delta q_i}, \frac{\delta f}{\delta p_i} \right) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta g}{\delta q_i} \\ \frac{\delta g}{\delta p_i} \end{pmatrix}^t, \tag{B.6}$$

where I and the superscript t are the $N \times N$ unit matrix and the transposition, respectively. A Casimir invariant C is defined so that $\{F, C\} = 0$ for any F , and is a conserved quantity since $dC/dt = \{H, C\} = 0$. Thus, if C is a Casimir invariant, we have

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta C}{\delta q_i} \\ \frac{\delta C}{\delta p_i} \end{pmatrix}^t = 0, \tag{B.7}$$

which tells that C does not depend on any of \mathbf{q} and \mathbf{p} , i.e., that C is trivial.

Because Eqs. (4.19) and (4.20) can be rewritten by means of the symplectic matrix, our Hamiltonian formulation given in §4.2 is canonical.

Appendix C

— Formulation for a Relativistic Perfect Fluid —

Let us consider a relativistic perfect fluid in a four-dimensional space-time region Ω . We redefine ρ and s as the particle number density and the entropy per particle in the local rest frame of matter, respectively, and define \mathbf{u} as the four-velocity field.

We assume that metric tensor $g(,)$ is given by other materials, and put the speed of light equal to unity. The normalization of the four-velocity field \mathbf{u} is given by

$$g(\mathbf{u}, \mathbf{u}) + 1 = 0 . \quad (\text{C}\cdot 1)$$

The constraints Eqs. (2·12), (2·13), and (3·9) are respectively rewritten as

$$L_{\mathbf{u}}(\rho * 1) = 0, \quad L_{\mathbf{u}}s = 0, \quad \text{and} \quad L_{\mathbf{u}}A_{\alpha} = 0, \quad \text{for} \quad \alpha = 1, 2 . \quad (\text{C}\cdot 2)$$

The Lagrangian density for a relativistic perfect fluid is given by

$$\mathcal{L}(\rho, s) = -\rho\epsilon . \quad (\text{C}\cdot 3)$$

Let $*1$ denote the four-dimensional volume element, and we can modify the action Eq. (3·10) as

$$S(\rho, s, \mathbf{A}, \kappa, \lambda, \boldsymbol{\beta}, \mathbf{u}, \gamma) = \int_{\Omega} \left\{ *1\mathcal{L}(\rho, s) + \gamma *1 \{g(\mathbf{u}, \mathbf{u}) + 1\} - \kappa L_{\mathbf{u}}(\rho * 1) - \rho\lambda *1L_{\mathbf{u}}s - \sum_{\alpha=1}^2 \rho\beta_{\alpha} *1L_{\mathbf{u}}A_{\alpha} \right\} , \quad (\text{C}\cdot 4)$$

where γ is an undetermined multiplier for the normalization. We still have Eq. (2·11), and impose restrictions Eqs. (2·15), (2·16), and (3·7) in the variational calculus. The stationary conditions corresponding to Eqs. (2·18), and (2·19) are respectively given by

$$-h + L_{\mathbf{u}}\kappa = 0 , \quad (\text{C}\cdot 5)$$

$$-T + L_{\mathbf{u}}\lambda = 0 , \quad (\text{C}\cdot 6)$$

$$L_{\mathbf{u}}\beta_{\alpha} = 0 , \quad (\text{C}\cdot 7)$$

$$2\gamma g(\mathbf{u},) + d\kappa - \lambda ds - \sum_{\alpha=1}^2 \beta_{\alpha} dA_{\alpha} = \mathbf{0} , \quad (\text{C}\cdot 8)$$

where enthalpy h , pressure p , and temperature T are defined in the local rest frame of matter. From Eqs. (C·1), (C·2), (C·5)–(C·8), we obtain $2\gamma = h$ and thus have the equation for the four-momentum field

$$hg(\mathbf{u},) + d\kappa - \lambda ds - \sum_{i=1}^2 \beta_{\alpha} dA_{\alpha} = \mathbf{0} , \quad (\text{C}\cdot 9)$$

of which the Lie derivative gives the Euler equation

$$L_{\mathbf{u}} \{hg(\mathbf{u},)\} + dp/\rho = \mathbf{0} . \quad (\text{C}\cdot 10)$$

Defining

$$\mathbf{v} \equiv \left(\frac{u_1}{u_0}, \frac{u_2}{u_0}, \frac{u_3}{u_0} \right) , \quad (\text{C}\cdot 11)$$

we have $u_0 = 1/\sqrt{1 - \mathbf{v}^2}$ from Eq. (C.1). As in §4.2, we take $\bar{\rho} \equiv \rho u_0$, s , and \mathbf{A} to be the state \mathbf{q} , take $-\kappa$, $-\rho\lambda$ and $-\rho\boldsymbol{\beta}$ to be the costate \mathbf{p} , and take \mathbf{v} to be the input. Because Eq. (C.2) gives

$$\frac{\partial}{\partial t} \bar{\rho} = -\nabla \cdot (\bar{\rho} \mathbf{v}) , \tag{C.12}$$

$$\frac{\partial}{\partial t} s = -\mathbf{v} \cdot \nabla s , \tag{C.13}$$

$$\frac{\partial}{\partial t} A_\alpha = -\mathbf{v} \cdot \nabla A_\alpha , \tag{C.14}$$

we can find the function corresponding to Eq. (4.14) to be given by

$$\mathcal{H}(\rho, s, \mathbf{A}, \lambda, \kappa, \boldsymbol{\beta}, \mathbf{v}) = -\mathcal{L}(\bar{\rho}\sqrt{1 - \mathbf{v}^2}, s) + \kappa \nabla \cdot (\bar{\rho} \mathbf{v}) + \bar{\rho} \lambda \mathbf{v} \cdot \nabla s + \sum_{i=\alpha}^2 \bar{\rho} \beta_\alpha \mathbf{v} \cdot \nabla A_\alpha , \tag{C.15}$$

and the optimized input $\mathbf{v}^*(\mathbf{q}, \mathbf{p})$ to satisfy Eq. (C.9).

References

- 1) A. Bennett, *Lagrangian fluid dynamics* (Cambridge Univ. Press, Cambridge, 2006), p. 32.
- 2) H. Bateman, Proc. R. Soc. London A **125** (1929), 598; Scripta Math. **10** (1944), 51; *Partial differential equations* (Dover, New York, 1944), p. 164.
- 3) A. Clebsch, J. Reine Angew. Math. **56** (1859), 1.
- 4) H. Lamb, *Hydrodynamics* (Cambridge Univ. Press, Cambridge, 1932), p. 248.
- 5) C. C. Lin, in *International School of Physics Enrico Fermi (XXI)*, ed. G. Careri (Academic Press, New York, 1963), p. 93.
- 6) R. L. Selinger and G. B. Whitham, Proc. R. Soc. London A **305** (1968), 1.
- 7) B. F. Schutz, Jr, Phys. Rev. D **2** (1970), 2762; Phys. Rev. D **4** (1971), 3559.
- 8) H.-T. Elze, Y. Hama, T. Kodama, M. Makler and J. Rafelski, J. of Phys. G **25** (1999), 1935.
- 9) J. L. Friedman and J. R. Ipser, Philos. Trans. R. Soc. London A **340** (1992), 39.
- 10) T. Kambe, Fluid Dyn. Res. **39** (2007), 98; Fluid Dyn. Res. **40** (2008), 399; Physica D **237** (2008), 2067; *Geometrical theory of dynamical systems and velocity field*, Revised ed. (World Scientific, Singapore, 2010), p. 189.
- 11) Z. Yoshida, in *Proc. Int. Symp., Contemporary Physics*, ed. J. Aslam, F. Hussain and Riazuddin (World Scientific, Singapore, 2008), p. 125.
- 12) Z. Yoshida, J. Math. Phys. **50** (2009), 113101.
- 13) D. D. Holm and B. A. Kupershmidt, Physica D **7** (1983), 330.
- 14) H. Cendra and J. E. Marsden, Physica D **27** (1987), 63.
- 15) D. D. Holm, J. E. Marsden and T. S. Ratiu, Adv. Math. **137** (1998), 1.
- 16) V. I. Arnold and B. A. Khesin, *Topological methods in hydrodynamics* (Springer-Verlag, Berlin, 1997), p. 47.
- 17) P. J. Morrison and J. M. Greene, Phys. Rev. Lett. **45** (1980), 790; Phys. Rev. E **48** (1982), 569.
- 18) R. Salmon, Ann. Rev. Fluid Mech. **20** (1988), 225.
- 19) L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mischenko, *The mathematical theory of optimal processes* (Gordon & Breach Science Pub., New York, 1987), p. 66.
- 20) M. Schulz, *Control theory in physics and other fields of science* (Springer-Verlag, Berlin, 2006), p. 17.
- 21) B. F. Schutz, *Geometrical methods of mathematical physics* (Cambridge Univ. Press, Cambridge, 1980), p. 181.