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Contributions of the Discrete Spectrum of the Maass Laplacians to Super Traces of Laplace Operators on Super Riemann Surfaces

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We study contributions of the discrete spectrum of the Maass laplacians to super traces of laplace operators on super Riemann surfaces. We show that the discrete spectrum makes a contribution only through the unit element of the super Fuchsian group in the Selberg super trace formula.

In the Polyakov formulation,[1] the closed (super)string measure for genus $g\geqq 2$ (super) Riemann surface $(s)M_g$ is given by an integral over (super) moduli space with respect to the (super) Weil-Petersson measure. The integrand consists of a certain combination of (super) determinants of laplacians on the surfaces.[2)~4)] D'Hoker and Phong evaluated these determinants in terms of the Selberg zeta functions. Furthermore, Baranov et al.[5] showed that the Selberg super zeta functions participate in the case of superstring. Aoki[6] computed the super determinants including overall factors by different method.

Aoki constructed heat kernels of laplacians on the super upper half plane sH by solving simultaneous partial differential equations. A complete set of eigenfunctions for the Maass laplacians[7] was needed to solve the equations. However, the eigenfunctions Aoki made use of do not form a complete set, because the discrete spectrum[7,8] is not taken into account.

It is reported recently[9] that the correct expression for the heat kernel of the Maass laplacians on the upper half plane H is

$$g_n{}^t(r)=\frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}}\int_r^\infty \frac{du\,ue^{-u^2/(4t)}}{\sqrt{\mathrm{ch}u-\mathrm{ch}r}}T_{2n}\left(\frac{\mathrm{ch}\frac{u}{2}}{\mathrm{ch}\frac{r}{2}}\right).$$

At first sight it may appear that the discrete spectrum does not contribute to the heat kernel, but this is not the case. In fact, the contribution of the discrete spectrum is already included in the above expression of $g_n{}^t(r)$.[10]

In the case of superstring contributions from the discrete spectrum to super traces are not sufficiently investigated yet. Our purpose is to study contributions from the discrete spectrum using explicit eigenfunctions.

The completeness relation we adopt is the following formula on H:[7,11]

$$\frac{1}{2\pi}\delta(\mathrm{ch}d-1)=\sum_{\{m\}}\frac{2|n|-2m-1}{4\pi}\left(\frac{z_1-\bar{z}_2}{z_2-\bar{z}_1}\right)^n P_{|n|-m,n}(\mathrm{ch}d)$$
$$+\int_{\mathrm{Re}\,s=1/2}ds\frac{(s-1/2)\sin 2\pi s}{8\pi i\sin\pi(s-n)\sin\pi(s+n)}\left(\frac{z_1-\bar{z}_2}{z_2-\bar{z}_1}\right)^n P_{s,n}(\mathrm{ch}d)\,,\quad(1)$$

where $\mathrm{ch}d=\mathrm{ch}d(z_1,z_2)$ is the hyperbolic distance between z_1 and z_2 on H, $\{m\}=\{m|m\in\boldsymbol{Z},\ 0\leqq m<|n|-1/2\}$ and the P's are given by the hyper geometric function F;

especially for the P in the first term, which corresponds to the discrete spectrum, we have

$$P_{|n|-m,n}(\mathrm{ch}d)=\left(\frac{2}{1+\mathrm{ch}d}\right)^{|n|-m}F\left(2|n|-m,\,-m,1;\frac{\mathrm{ch}d-1}{\mathrm{ch}d+1}\right)$$
$$=\frac{(-1)^m\Gamma(2|n|-m)}{m!\Gamma(2|n|-2m)}\left(\frac{2}{1+\mathrm{ch}d}\right)^{|n|-m}$$
$$\times F\left(2|n|-m,\,-m,2|n|-2m;\frac{2}{1+\mathrm{ch}d}\right). \tag{2}$$

[$g_n^t(r)$ is derived from the second term of Eq. (1) with the line of integral moved from Re$s=1/2$ to Re$s>|n|$ or Re$s<1-|n|$. The second term of Eq. (1) alone does not lead to the above $g_n^t(r)$. The first term fills the role of moving the line of integral as above.[10]] We pay attention to contributions of the first term in Eq. (1) hereafter.

We introduce Aoki's argument briefly. We consider the following laplacian:

$$\Delta_n=z_{1\bar{1}}{}^2\partial_z\partial_{\bar{z}}+z_{1\bar{1}}\theta_{1\bar{1}}(\partial_z\hat{D}_-+\partial_{\bar{z}}\hat{D}_+)+(2n+1)z_{1\bar{1}}\hat{D}_-\hat{D}_+$$
$$-nz_{1\bar{1}}(\partial_z+\partial_{\bar{z}})+2n^2\theta_{1\bar{1}}(\hat{D}_-+\hat{D}_+)-n^2, \tag{3}$$

where

$$z_{ab}=z_a-z_b-\theta_a\theta_b, \qquad \theta_{ab}=\theta_a-\theta_b,$$
$$\hat{D}_+=\partial_\theta-\theta\partial_z, \qquad \hat{D}_-=\partial_{\bar{\theta}}-\bar{\theta}\partial_{\bar{z}}.$$

In Eq. (3) all derivatives act on variables with suffix 1. The heat kernel for the invariant operator $(z_{1\bar{2}}/z_{2\bar{1}})^{-n}\Delta_n(z_{1\bar{2}}/z_{2\bar{1}})^n$ can be written as

$$G_n^t(Z_1,Z_2)=g_n^t(r)+i\Delta\bar{\Delta}h_n^t(r), \tag{4}$$

where $Z=(z,\theta)$, and r, Δ, $\bar{\Delta}$ are 1 Grassmann even and 2 odd invariants:

$$\mathrm{ch}r(Z_1,Z_2)=1-2\frac{z_{12}z_{\bar{1}\bar{2}}}{z_{1\bar{1}}z_{2\bar{2}}},$$
$$\Delta=\frac{\theta_1z_{2\bar{2}}+\theta_2z_{\bar{2}1}+\theta_{\bar{2}}z_{12}+\theta_1\theta_2\theta_{\bar{2}}}{(z_{12}z_{2\bar{2}}z_{\bar{2}1})^{1/2}}.$$

$g_n^t(r)$ and $h_n^t(r)$ should obey the following simultaneous partial differential equations:

$$\left\{\partial_t-\left[\partial_r{}^2+\left(\frac{1}{\mathrm{sh}r}-2n\mathrm{th}\frac{r}{2}\right)\partial_r+n^2\left(1+\frac{2}{\mathrm{ch}r+1}\right)\right]\right\}g_n^t(r)=\frac{2(2n+1)}{\mathrm{sh}r}h_n^t(r), \tag{5}$$

$$\left\{\partial_t-\left[\partial_r{}^2+\left(-\frac{1}{\mathrm{sh}r}+2n\mathrm{th}\frac{r}{2}\right)\partial_r+\frac{\mathrm{ch}r}{\mathrm{sh}^2r}+\frac{2(n+n^2)}{\mathrm{ch}r+1}+n^2\right]\right\}h_n^t(r)$$
$$=2n\frac{\mathrm{sh}r}{\mathrm{ch}r+1}\left[\partial_r{}^2+\frac{1}{\mathrm{sh}r}\partial_r+\frac{n}{\mathrm{ch}r+1}+n^2\right]g_n^t(r) \tag{6}$$

with the condition

$$g_n{}^t(r) \xrightarrow{t\to 0} 0\,, \tag{7}$$

$$\frac{h_n{}^t(r)}{\operatorname{sh} r} \xrightarrow{t\to 0} -\frac{1}{2\pi}\delta(\operatorname{ch} r-1)\,. \tag{8}$$

[Equation (6) is partly different from Eq. (3·7) in Ref. 6) in signs.] Aoki's idea is that one can deduce $g_n{}^t(r)$ from $\hat{g}_n{}^t(z_1, z_2)$ which is the solution of

$$\begin{aligned}&\{[\partial_t-(D_{-n}+(n+1)^2)][\partial_t-(D_{-n}+n^2)]-(2n+1)^2D_{-n}\}\,\hat{g}_n{}^t(z_1, z_2)\\ &\quad=-(2n+1)y_1{}^2\delta(x_1-x_2)\delta(y_1-y_2)\,,\end{aligned} \tag{9}$$

where

$$D_{-n}=y^2(\partial_x{}^2+\partial_y{}^2)+2iny\partial_x$$

is the Maass laplacian; $g_n{}^t(r)$ and $\hat{g}_n{}^t(z_1, z_2)$ are related as

$$\text{the body of } g_n{}^t(r)=\left(\frac{z_1-\bar{z}_2}{z_2-\bar{z}_1}\right)^{-n}\hat{g}_n{}^t(z_1, z_2)\,.$$

We split $g_n{}^t(r)$ and $h_n{}^t(r)$ into two parts:

$$g_n{}^t(r)=g_n^{t,d}(r)+g_n^{t,c}(r)\,,$$

$$h_n{}^t(r)=h_n^{t,d}(r)+h_n^{t,c}(r)\,,$$

where d and c denote contributions of the discrete spectrum and that of the continuous spectrum respectively. Aoki has found $g_n^{t,c}(r)$ and $h_n^{t,c}(r)$ that satisfy Eqs. (5) and (6). Let us find $g_n^{t,d}(r)$ and $h_n^{t,d}(r)$. We can express the δ-functions in Eq. (9) in terms of the P's as in Eq. (1). The first term on the r.h.s. of Eq. (1) contributes to $\hat{g}_n^{t,d}(z_1, z_2)$, from which $g_n^{t,d}(r)$ can be deduced. To solve Eq. (9) we use the laplace transform in t, and we deduce

$$g_n^{t,d}(r)=\begin{cases}\dfrac{1}{4\pi}\sum\limits_{\{m\}}(e^{t(m+1)^2}-e^{t(2|n|-m)^2})P_{|n|-m,n}(\operatorname{ch} r)\,, & (n>0)\\[2ex] -\dfrac{1}{4\pi}\sum\limits_{\{m\}}(e^{tm^2}-e^{t(2|n|-m-1)^2})P_{|n|-m,n}(\operatorname{ch} r)\,. & (n<0)\end{cases} \tag{10}$$

Equation (5) should also hold for $g_n^{t,d}(r)$ and $h_n^{t,d}(r)$, and we find

$$h_n^{t,d}(r)=\begin{cases}-\dfrac{\operatorname{sh} r}{8\pi}\sum\limits_{\{m\}}((|n|-m-1)e^{t(m+1)^2}+(|n|-m)e^{t(2|n|-m)^2})P_{|n|-m,n}(\operatorname{ch} r)\\[1ex] \quad+\dfrac{\operatorname{ch} r-1}{8\pi}\sum\limits_{\{m\}}(e^{t(m+1)^2}-e^{t(2|n|-m)^2})\partial_r P_{|n|-m,n}(\operatorname{ch} r)\,, \quad (n>0)\\[2ex] -\dfrac{\operatorname{sh} r}{8\pi}\sum\limits_{\{m\}}((|n|-m)e^{tm^2}+(|n|-m-1)e^{t(2|n|-m-1)^2})P_{|n|-m,n}(\operatorname{ch} r)\\[1ex] \quad-\dfrac{\operatorname{ch} r-1}{8\pi}\sum\limits_{\{m\}}(e^{tm^2}-e^{t(2|n|-m-1)^2})\partial_r P_{|n|-m,n}(\operatorname{ch} r)\,. \quad (n<0)\end{cases} \tag{11}$$

These $g_n^{t,d}(r)$ and $h_n^{t,d}(r)$ satisfy Eq. (6) as well. As for the initial condition we have

$$g_n^{t,d}(r)\xrightarrow{t\to 0}0\,,$$

$$\frac{h_n^{t,d}(r)}{\mathrm{sh}r}\xrightarrow{t\to 0}-\frac{1}{8\pi}\sum_{\{m\}}(2|n|-2m-1)P_{|n|-m,n}(\mathrm{ch}r)\,.$$

The former comes up to our expectations. The latter is half of our naive expectation, if we compare Eqs. (1) and (8). However, it may be possible that the rest half is supplied from $h_n^{t,c}(r)/\mathrm{sh}r$.[10)]

The heat kernel for Δ_n on the super Riemann surface $sM_g=sH/s\Gamma$ can be constructed by the Poincaré sum over the super Fuchsian group $s\Gamma$:

$$\mathrm{Str}e^{-t\Delta_n}=I_n^e(t)+I_n(t)\,, \tag{12}$$

where

$$\begin{aligned}I_n^e(t)&=I_n^{e,d}(t)+I_n^{e,c}(t)\\&=\int_{sM_g}d^2zd^2\theta\frac{-4}{z_{1\bar{1}}}G_n^t(Z,Z)\,,\end{aligned} \tag{13}$$

$$\begin{aligned}I_n(t)&=I_n^d(t)+I_n^c(t)\\&=\sum_{\substack{\gamma\in s\Gamma\\ \gamma\neq e}}\int_{sM_g}d^2zd^2\theta\frac{-4}{z_{1\bar{1}}}\left(\frac{c\bar{z}+d+\beta\bar{\theta}}{cz+d+\beta\theta}\right)^n\left(\frac{z-\gamma\bar{z}-\theta\gamma\bar{\theta}}{\gamma z-\bar{z}-\gamma\theta\bar{\theta}}\right)^n G_n^t(Z,\gamma Z)\end{aligned} \tag{14}$$

with

$$s\Gamma\ni\gamma=\begin{pmatrix}a & b & \alpha\\ c & d & \beta\\ \delta & \epsilon & \nu_\gamma(1-\alpha\beta)\end{pmatrix},\qquad \nu_\gamma=\pm 1\,.$$

In the computation of the super determinant of Δ_n, $I_n(t)$ and $I_n^e(t)$ correspond to the Selberg super zeta functions and an overall factor respectively. Let us calculate $I_n^{e,d}(t)$ and $I_n^d(t)$.

As for $I_n^{e,d}(t)$, we can carry out the integration easily to obtain

$$I_n^{e,d}(t)=\begin{cases}\dfrac{|\chi(M_g)|}{2}\sum\limits_{\{m\}}(e^{t(m+1)^2}-e^{t(2|n|-m)^2})\,, & (n>0)\\ -\dfrac{|\chi(M_g)|}{2}\sum\limits_{\{m\}}(e^{tm^2}-e^{t(2|n|-m-1)^2})\,, & (n<0)\end{cases} \tag{15}$$

where $\chi(M_g)=2-2g(g\geqq 2)$ is the Euler characteristic of the underlying manifold M_g. Thus $I_n^{e,d}(t)$ is nonzero unless $|n|\leqq 1/2$.

According to Selberg,[12)] the summation over $s\Gamma$ in Eq. (14) can be reduced to the summation over primitive elements and their powers. Each of the primitive elements is characterized by an associated length l_γ. After the well-known change of varibles $(x,y)\to(v=x/y,y)$, the integrations over θ, $\bar{\theta}$ and y can be done and we find, for the $n>0$ case,

$$I_n^d(t)=\sum_{\gamma\text{prim}}\sum_{p=1}^{\infty}\nu_\gamma^{2np}l_\gamma\sum_{\{m\}}\frac{(-1)^m\Gamma(2|n|-m)}{4\pi m!\Gamma(2|n|-2m)}\int_{-\infty}^{\infty}dv\left(\frac{\text{ch}\frac{pl_\gamma}{2}+iv\text{sh}\frac{pl_\gamma}{2}}{\text{ch}\frac{pl_\gamma}{2}-iv\text{sh}\frac{pl_\gamma}{2}}\right)^n$$

$$\times\left[(e^{t(m+1)^2}-e^{t(2|n|-m)^2})+4(1+v^2)\text{sh}^2\frac{pl_\gamma}{2}(e^{t(m+1)^2}-e^{t(2|n|-m)^2})\frac{\partial}{\partial\text{ch}d}\right.$$

$$+2ni\nu_\gamma^p\text{sh}\frac{pl_\gamma}{2}\frac{v}{\text{ch}^2\frac{pl_\gamma}{2}+v^2\text{sh}^2\frac{pl_\gamma}{2}}(e^{t(m+1)^2}-e^{t(2|n|-m)^2})$$

$$-2\left(\nu_\gamma^p\text{ch}\frac{pl_\gamma}{2}-1\right)((|n|-m-1)e^{t(m+1)^2}+(|n|-m)e^{t(2|n|-m)^2})$$

$$\left.+2\left(\nu_\gamma^p\text{ch}\frac{pl_\gamma}{2}-1\right)(e^{t(m+1)^2}-e^{t(2|n|-m)^2})(\text{ch}d-1)\frac{\partial}{\partial\text{ch}d}\right]$$

$$\times\left(\frac{2}{1+\text{ch}d}\right)^{|n|-m}F\left(2|n|-m,\,-m,\,2|n|-2m;\,\frac{2}{1+\text{ch}d}\right),\tag{16}$$

where $\text{ch}d=\text{ch}d(z, e^{pl_\gamma}z)=\text{ch}pl_\gamma+2v^2\text{sh}^2((pl_\gamma)/2)$. Note that $F(2|n|-m, -m, 2|n|-2m; 2/(1+\text{ch}d))$ is a polynomial of degree $\leqq m$ in $(1+\text{ch}d)^{-1}$, and that

$$1+\text{ch}d=2\left(\text{ch}\frac{pl_\gamma}{2}+iv\text{sh}\frac{pl_\gamma}{2}\right)\left(\text{ch}\frac{pl_\gamma}{2}-iv\text{sh}\frac{pl_\gamma}{2}\right).$$

Naive power counting of v shows us that the integration over v in Eq. (16) can be treated by a contour integration. [It is convenient to choose the contour on the upper (lower) half plane for n positive (negative).] The first and fourth terms in the square brackets in Eq. (16) have no pole in the contour, and they vanish. This situation is analogous to the bosonic case.[13] While $(1+\text{ch}d)^{-m}$ in F gives rise to a pole of first order for the second, third and fifth terms in Eq. (16). Each of them produces the following nonzero value respectively:

$$\sum_{\gamma,p}J_{\gamma,p}\frac{1}{\text{ch}\frac{pl_\gamma}{2}},\quad -\sum_{\gamma,p}J_{\gamma,p}\nu_\gamma^p,\quad \sum_{\gamma,p}J_{\gamma,p}\left(-\frac{1}{\text{ch}\frac{pl_\gamma}{2}}+\nu_\gamma^p\right),$$

where

$$J_{\gamma,p}=\sum_{\{m\}}\nu_\gamma^{2np}l_\gamma(e^{t(m+1)^2}-e^{t(2|n|-m)^2})\frac{|n|\Gamma(2|n|)}{2^{2|n|+1}\text{sh}\frac{pl_\gamma}{2}\text{ch}^{2|n|}\frac{pl_\gamma}{2}m!\Gamma(2|n|-m)}.$$

However, they successfully sum up to zero. Thus we find $I_n^d(t)=0$ for $n>0$. It is easy to get the same result for $n<0$.

Our conclusion is that the discrete spectrum contributes to the super traces only through the unit element of the super Fuchsian group $s\Gamma$ in the Selberg super trace formula. Our result is a super-analogue of the result by D'Hoker and Phong.[13]

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