Canonical Formulation of Mixed State and Irreducible Representation of $u(\mathcal{M})$ Algebra

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In order to get an irreducible representation of $u(\mathcal{M})$ algebra, we develop a canonical theory of mixed state and discuss the dual property of mutually commutable $u(\mathcal{M})$ algebras which have the Casimir operators in common. With the aid of this dual property, we can obtain an irreducible representation of $u(\mathcal{M})$ algebra which is useful for investigations of nuclear dynamics, where generators of $u(\mathcal{M})$ algebra given by one-body operators of fermion are expressed in terms of bosons. As an example, we derive the Holstein-Primakoff representation of $u(\mathcal{M})$ generators. We also derive the irreducible representation of $su(3)$ algebra.

§1. Introduction

In the microscopic description of nuclear collective motions, schematic models based on various dynamical groups have played an important role. Main attention has been paid to the ground and low-excited collective band structures. Hence the adopted space of irreducible representation of the dynamical group has been greatly restricted. However, recent interests have been attracted to studying the higher-excited collective band and also the broken symmetry in association with the transition from an ordered phase to a disordered or chaotic phase. Hence it becomes necessary to develop a simple and systematic prescription of the irreducible representation for more vast numerical evaluation.

In this paper, we deal with the $u(\mathcal{M})$ algebra, which is introduced in association with a nuclear model with $\mathcal{M}$ single-particle levels of the same degeneracy.*) In order to get an irreducible representation favorable for numerical evaluations, we apply the canonical formulation of mixed state, which has been proposed as a canonical formulation of the thermo field dynamics.[1] In §2, we give an atomic nuclear model generated by a $u(\mathcal{M})$ algebra and briefly recapitulate the canonical formulation of mixed state.[2] As an expectation value with respect to the mixed state, we define a classical image of one-body operators of fermion, which satisfies the $u(\mathcal{M})$ algebra described in terms of the Poisson bracket. In this formulation, a set of constraints is introduced for guaranteeing the unitary nature of the generator of the mixed state. In §3, the algebraic structure of the constraints is discussed by following the viewpoint developed in Ref. 3). That is, the constraints are regarded as constructing another $u(\mathcal{M})$ algebra. Following this viewpoint, we make clear the dual property of

*) Results on the irreducible representation given here is purely mathematical and independent of the atomic nuclear model. However, the present treatment of $u(\mathcal{M})$ algebra will make clear the physical image or the physical ingredient of the irreducible representation.

the $u(\mathcal{M})$ algebra and the algebra prescribed by the constraints, which are mutually commutable and have common Casimir operators. In §4, we show how to work with this duality for obtaining the basis of the irreducible representation carrier space of the $u(\mathcal{M})$ algebra, where we adopt the canonical quantization. Section 5 is devoted to deriving the irreducible representation which is useful for investigating the subject mentioned at the beginning. In this derivation, the duality and also the retractable property of algebra will play an important role. As an example, we take up the irreducible representation of $u(\mathcal{M})$ algebra which can be retracted to a $u(2)$ or a $u(1)$ algebra. In §6, we give a $u(2)$ algebra in relation to the Lipkin model as an illustration of how to work with the present approach. In §7, we deal with the $u(3)$ algebra. We show that we can easily derive the irreducible representation of $su(3)$ with the use of 3-j and 6-j symbols, and make clear the relation to the conventional approach.

§2. Canonical formulation of mixed state

In this section, we introduce a $u(\mathcal{M})$ algebra associated with an atomic nuclear model and briefly recapitulate the canonical formulation of mixed state, which has been developed in Refs. 2) and 3).

Let us take up a nucleon system of $\mathcal{M}$ single-particle levels with the same degeneracy $2j+1(=2\Omega)$. The single-particle states are specified in terms of the set of parameters, $\{(p,j,m);\ p=1,2,\cdots,\mathcal{M},\ m=-j,-j+1,\cdots,j\}$. The nucleon creation operator in the single-particle state (p,j,m) is denoted by $\hat{c}^\dagger_{pjm}$. The hamiltonian is given as follows:

$$\hat{H}=\sum_{p=1}^{\mathcal{M}}\epsilon_p\sum_{m=-j}^{j}\hat{c}^\dagger_{pjm}\hat{c}_{pjm}-\frac{1}{2}\sum_{p,q,p',q'=1}^{\mathcal{M}}V_{pq:p'q'}\sum_{m=-j}^{j}\hat{c}^\dagger_{pjm}\hat{c}_{qjm}\sum_{m=-j}^{j}\hat{c}^\dagger_{p'jm}\hat{c}_{q'jm}. \tag{2·1}$$

Here we define the generators of $u(\mathcal{M})$ algebra in terms of one-body operators of fermion:

$$\hat{E}^p_q=\sum_{m=-j}^{j}\hat{c}^\dagger_{pjm}\hat{c}_{qjm}, \tag{2·2}$$

which satisfy the following $u(\mathcal{M})$ algebra:*)

$$[\hat{E}^p_q,\hat{E}^r_s]_-=\delta_{rq}\hat{E}^p_s-\delta_{ps}\hat{E}^r_q. \tag{2·3}$$

Therefore, we can treat the above hamiltonian with the aid of the irreducible representation of $u(\mathcal{M})$ algebra. The main aim of this paper is to obtain a description of

*) The conservation of the total nucleon number implies that we can eliminate one of the generators, resulting in an $su(\mathcal{M})$ algebra. For instance, the generators of the $su(\mathcal{M})$ algebra can be given as

$$\hat{\tilde{E}}^p_q=\hat{E}^p_q-\delta_{pq}\frac{1}{\mathcal{M}}\hat{N},\quad\left(\hat{N}=\sum_p\hat{E}^p_p\right)$$

where $\hat{N}$ denotes the total nucleon number. In this sense, $\hat{E}^p_q$ generates the $su(\mathcal{M})$ algebra.

the irreducible representation in a favorable form for a systematical and numerical evaluation, and to derive a bosonic representation of the above one-body operators of fermion.

2.1. *Mixed state*

Following Ref. 1), we introduce an auxiliary fermion field, the state of which is specified in terms of the same labels as the nucleon field. They satisfy the following anti-commutation relations:

$$\begin{aligned}
&[\hat{c}_{pjm}, \hat{c}^{\dagger}_{p'jm'}]_+ = \delta_{pp'}\delta_{mm'}, \quad [\hat{c}_{pjm}, \hat{c}_{p'jm'}]_+ = 0, \\
&[\hat{d}_{pjm}, \hat{d}^{\dagger}_{p'jm'}]_+ = \delta_{pp'}\delta_{mm'}, \quad [\hat{d}_{pjm}, \hat{d}_{p'jm'}]_+ = 0, \\
&[\hat{c}_{pjm}, \hat{d}_{p'jm'}]_+ = [\hat{c}_{pjm}, \hat{d}^{\dagger}_{p'jm'}]_+ = 0.
\end{aligned} \tag{2·4}$$

The vacuum of mixed state is given by

$$|0\rangle\rangle = V|0\rangle_c|0\rangle_d, \quad (\hat{c}_{pjm}|0\rangle_c = \hat{d}_{pjm}|0\rangle_d = 0) \tag{2·5a}$$

$$V = \exp\left[\sum_p \theta_p \sum_m (\hat{c}^{\dagger}_{pjm}\hat{d}^{\dagger}_{\widetilde{pjm}} - \hat{d}_{\widetilde{pjm}}\hat{c}_{pjm})\right], \tag{2·5b}$$

where the variables θ_p are real and $\hat{d}^{\dagger}_{\widetilde{pjm}}$ expresses the creation operator in the time-reversal state, i.e., $(-1)^{j-m}\hat{d}^{\dagger}_{pj-m}$.

Then the expectation values of one-body operators with respect to the vacuum $|0\rangle\rangle$ are given as follows:

$$\begin{aligned}
\langle c^{\dagger}_{pjm}c_{p'jm'}\rangle_0 &= \langle\langle 0|\hat{c}^{\dagger}_{pjm}\hat{c}_{p'jm}|0\rangle\rangle \\
&= \delta_{pp'}\delta_{mm'}v_p^2, && (2·6a) \\
\langle d^{\dagger}_{pjm}d_{p'jm'}\rangle_0 &= \langle\langle 0|\hat{d}^{\dagger}_{pjm}\hat{d}_{p'jm'}|0\rangle\rangle \\
&= \delta_{pp'}\delta_{mm'}v_p^2, && (2·6b) \\
v_p &= \sin\theta_p. && (2·6c)
\end{aligned}$$

From the above relations, it is apparent that the quantity v_p^2 implies the occupancy of the single-particle state (p, j, m) in the vacuum $|0\rangle\rangle$.

Now let us define the mixed state as

$$|m\rangle\rangle = U|0\rangle\rangle, \tag{2·7a}$$

$$U = \exp\left[i\sum_{p,q} \Gamma_{pq}\hat{E}^p_q\right], \tag{2·7b}$$

where the matrix Γ, the element of which is given by Γ_{pq}, is hermite, that is

$$\Gamma^{\dagger} = \Gamma. \tag{2·8}$$

2.2. *Classical image*

The classical images of one-body operators are defined by their expectation values with respect to the mixed state as follows:

$$\begin{aligned}\langle c^\dagger_{pjm} c_{qjm'}\rangle_m &= \langle\langle m|\hat{c}^\dagger_{pjm}\hat{c}_{qjm'}|m\rangle\rangle \\ &= \delta_{mm'}\sum_a v_a^2 u_{ap}u^*_{aq}, \end{aligned} \tag{2·9a}$$

$$\begin{aligned}\langle d^\dagger_{pjm} d_{qjm'}\rangle_m &= \langle\langle m|\hat{d}^\dagger_{pjm}\hat{d}_{qjm'}|m\rangle\rangle \\ &= \delta_{pq}\delta_{mm'}v_p^2. \end{aligned} \tag{2·9b}$$

In the above relations, the unitary matrix u is defined as

$$u = \exp[-i\Gamma], \tag{2·10a}$$

$$\sum_p u_{ap}u^*_{bp} = \delta_{ab}, \tag{2·10b}$$

where the superscript '$*$' denotes the complex conjugation.

Then the classical image of generators of $u(\mathcal{M})$ algebra is given in the following form:

$$\begin{aligned}E^p_q &= \langle\langle m|\hat{E}^p_q|m\rangle\rangle \\ &= \sum_a n_a u_{ap}u^*_{aq}, \end{aligned} \tag{2·11}$$

$$n_a = (2j+1)v_a^2, \tag{2·12}$$

where n_a expresses the occupation number in the single-particle level a in the vacuum $|0\rangle\rangle$.

2.3. *Canonicity condition*

In the above definition of the mixed state, we have introduced a set of $\mathcal{M}(\mathcal{M}-1)+2\mathcal{M}$ variables, $\{(\Gamma_{pq}, \Gamma^*_{pq}; 1 \le p < q \le \mathcal{M}), (\Gamma_{pp}, \theta_p; 1 \le p \le \mathcal{M})\}$. In order to transform the $u(\mathcal{M})$ algebra given by Eq. (2·3) into a corresponding classical one, it is necessary to express the classical image (2·11) in terms of a set of canonical variables. For this aim, we introduce the following type of the canonicity condition:[4),5)]

$$\langle\langle m|\partial_x|m\rangle\rangle = \frac{1}{2}x^*, \quad \langle\langle m|\partial_{x^*}|m\rangle\rangle = -\frac{1}{2}x. \tag{2·13}$$

For the present case, the above condition can be explicitly written as

$$\begin{aligned}\langle\langle m|\partial_x|m\rangle\rangle &= \frac{1}{2}\sum_{a,p} n_a(u_{ap}\cdot\partial_x u^*_{ap} - \partial_x u_{ap}\cdot u^*_{ap}) \\ &= \frac{1}{2}x^*, \\ \langle\langle m|\partial_{x^*}|m\rangle\rangle &= \frac{1}{2}\sum_{a,p} n_a(u_{ap}\cdot\partial_{x^*} u^*_{ap} - \partial_{x^*} u_{ap}\cdot u^*_{ap}) \\ &= -\frac{1}{2}x. \end{aligned} \tag{2·14}$$

In order to find the canonical variables, let us introduce the following set of variables:

$$X_{ap} = \sqrt{n_a}\exp[i\phi_a]u^*_{ap}, \quad X^*_{ap} = \sqrt{n_a}\exp[-i\phi_a]u_{ap}; \quad a \neq p, \tag{2·15a}$$

$$X_{aa} = X^*_{aa} = \sqrt{n_a - \sum_{p(\neq a)} X^*_{ap}X_{ap}}. \tag{2·15b}$$

When these variables are regarded as mutually independent, they satisfy the canonicity condition[5)]

$$\langle\langle m|\partial_{X_{ap}}|m\rangle\rangle = \frac{1}{2}X^*_{ap}, \quad \langle\langle m|\partial_{X^*_{ap}}|m\rangle\rangle = -\frac{1}{2}X_{ap}, \tag{2·16a}$$

$$\langle\langle m|i\partial_{\phi_a}|m\rangle\rangle = n_a, \quad \langle\langle m| - i\partial_{n_a}|m\rangle\rangle = 0. \tag{2·16b}$$

Therefore the set of $2\mathcal{M}(\mathcal{M}-1)+2\mathcal{M}$ variables, $\{(X_{ap}, X^*_{ap}; 1 \leq a \neq p \leq \mathcal{M}), (\phi_a, n_a; 1 \leq a \leq \mathcal{M})\}$, can be regarded as canonical as long as they are mutually independent.

Then the Poisson bracket can be defined as follows:

$$\begin{aligned}\{A, B\}_0 = &\sum_{a\neq p}(\partial_{X_{ap}}A \cdot \partial_{X^*_{ap}}B - \partial_{X^*_{ap}}A \cdot \partial_{X_{ap}}B) \\ &+ i\sum_a(\partial_{\phi_a}A \cdot \partial_{n_a}B - \partial_{n_a}A \cdot \partial_{\phi_a}B).\end{aligned} \tag{2·17}$$

The canonical variables satisfy the relations

$$\{X_{ap}, X^*_{bq}\}_0 = \delta_{ab}\delta_{pq}, \quad \{X_{ap}, X_{bq}\}_0 = 0, \tag{2·18a}$$

$$\{\phi_a, n_b\}_0 = i\delta_{ab}, \quad \{\phi_a, \phi_b\}_0 = \{n_a, n_b\}_0 = 0, \tag{2·18b}$$

$$\{X_{ap}, \phi_b\}_0 = \{X_{ap}, n_b\}_0 = 0. \tag{2·18c}$$

The classical image (2·11) can be rewritten in the following form:

$$\begin{aligned}E^p_q &= \sum_a X^*_{ap}X_{aq} \\ &= \begin{cases} \sum_{a(\neq p,q)} X^*_{ap}X_{aq} + \sqrt{n_p - \sum_{r(\neq p)} X^*_{pr}X_{pr}}\ X_{pq} \\ \qquad + X^*_{qp}\sqrt{n_q - \sum_{r(\neq q)} X^*_{qr}X_{qr}}; & p \neq q, \\ \sum_{a(\neq p)} X^*_{ap}X_{ap} + (n_p - \sum_{r(\neq p)} X^*_{pr}X_{pr}); & p = q. \end{cases}\end{aligned} \tag{2·19}$$

The algebra can be transformed into the corresponding classical one with the use of the Poisson bracket:

$$\{E^p_q, E^r_s\}_0 = \delta_{rq}E^p_s - \delta_{ps}E^r_q. \tag{2·20}$$

In order to ascertain the above relation, it is convenient to introduce another set of variables $\{x_{ap}, x^*_{ap}; 1 \leq a, p \leq \mathcal{M}\}$, which are related with the old ones by the following relations:

$$(x_{ap}, x^*_{ap}) = (\exp[-i\phi_a]X_{ap}, X^*_{ap}\exp[i\phi_a]); \quad a \neq p, \tag{2·21a}$$

$$(x_{aa}, x^*_{aa}) = \left(\exp[-i\phi_a] \sqrt{n_a - \sum_{p(\neq a)} X^*_{ap} X_{ap}}\, , \sqrt{n_a - \sum_{p(\neq a)} X^*_{ap} X_{ap}} \exp[i\phi_a] \right). \tag{2·21b}$$

Since the new variables satisfy the relations

$$\{x_{ap}, x^*_{bq}\}_0 = \delta_{ab}\delta_{pq}, \quad \{x_{ap}, x_{bq}\}_0 = \{x^*_{ap}, x^*_{bq}\}_0 = 0, \tag{2·22}$$

the relation (2·21) defines a canonical transformation and the new set of variables becomes canonical. The Poisson bracket in terms of the new set of variables is given as

$$\{A, B\} = \sum_{ap} (\partial_{x_{ap}} A \cdot \partial_{x^*_{ap}} B - \partial_{x^*_{ap}} A \cdot \partial_{x_{ap}} B) \ (= \{A, B\}_0). \tag{2·23}$$

The classical image (2·19) can be rewritten in the following simple form:

$$E^p_q = \sum_a x^*_{ap} x_{aq}. \tag{2·24}$$

Then the commutation relation (2·20) can be easily ascertained. Hereafter we adopt the new set of canonical variables for the derivation of relations invariant under canonical transformations.

2.4. *Constraint*

In the preceding subsection, we have introduced a set of $2\mathcal{M}(\mathcal{M}-1)+2\mathcal{M}$ canonical variables. Since the number of the original independent variables is $\mathcal{M}(\mathcal{M}-1)+2\mathcal{M}$, these variables are not mutually independent. Hence there are necessary $\mathcal{M}(\mathcal{M}-1)$ relations among the present variables. These relations are provided with the aid of the unitary relation of the matrix u given by Eq. (2·10b). This relation can be written in terms of the present variables in the form

$$\sum_p x^*_{ap} x_{bp} = \begin{cases} 0 \ ; & a \neq b, \\ n_a \, ; & a = b. \end{cases} \tag{2·25}$$

The present selection of the canonical variables always satisfies the second relation. The first relation has been regarded as the $\mathcal{M}(\mathcal{M}-1)$ constraints on the canonical variables in Refs. 1) and 2).

However, the above relation has been taken up from a completely different viewpoint in Ref. 3). In succeeding sections, we investigate the structure of the constraints according to this viewpoint.

§3. Dual $u(\mathcal{M})$ algebra

In Ref. 3), it has been pointed out that the algebraic structure of the constraints may have a very significant ingredient.

3.1. *Fictitious algebra*

Following the viewpoint given in Ref. 3), we define a new set of bilinear forms in the following way:

$$F_b^a = \sum_p x_{ap}^* x_{bp}, \quad (a, b = 1, 2, \cdots, \mathcal{M}) \tag{3·1}$$

which also satisfy the $u(\mathcal{M})$ algebra in a classical version:*)

$$\{F_b^a, F_d^c\} = \delta_{bc} F_d^a - \delta_{ad} F_b^c. \tag{3·2}$$

This algebra is called a fictitious algebra and denoted by F-$u(\mathcal{M})$ hereafter. Its representation carrier space is also called the fictitious space. On the other hand, we call the algebra generated by E_q^p and its representation carrier space the E-$u(\mathcal{M})$ and the physical space, respectively, if necessary to distinguish them from the fictitious ones.

Taking advantage of the overcompleteness of the adopted set of canonical variables, we have introduced the F-$u(\mathcal{M})$ algebra. In this section, we investigate the algebraic structure and the relation between the E- and the F-$u(\mathcal{M})$ algebra in more details.

Here we summarize the results in the preceding discussions:

$$E_q^p = \sum_a x_{ap}^* x_{aq}, \quad F_b^a = \sum_p x_{ap}^* x_{bp}, \tag{3·3}$$

$$\{E_q^p, E_s^r\} = \delta_{rq} E_s^p - \delta_{ps} E_q^r, \quad \{F_b^a, F_d^c\} = \delta_{bc} F_d^a - \delta_{ad} F_b^c, \quad \{E_q^p, F_b^a\} = 0. \tag{3·4}$$

In this way, we have two mutually commutable $u(\mathcal{M})$ algebras described in terms of the same variables. The overcompleteness of the canonical variables enables us to construct such a couple of algebra, through by the unitarity of the matrix u.

3.2. *Dual $u(\mathcal{M})$ algebra*

The Casimir operators of the k-th order for the E- and the F-$u(\mathcal{M})$ algebra are given respectively by

$$C_E^{(k)} = \sum_{p_1, p_2, \cdots, p_k} E_{p_2}^{p_1} E_{p_3}^{p_2} \cdots E_{p_1}^{p_k}, \tag{3·5}$$

$$C_F^{(k)} = \sum_{a_1, a_2, \cdots, a_k} F_{a_2}^{a_1} F_{a_3}^{a_2} \cdots F_{a_1}^{a_k}. \tag{3·6}$$

Here we note that the two types of Casimir operators are identical to each other

$$C_E^{(k)} = C_F^{(k)}, \tag{3·7}$$

*) Since the summation of F_a^a is nothing but the total nucleon number in the classical version, i.e.,

$$\sum_a F_a^a = \sum_a \sum_p x_{ap}^* x_{ap} = \sum_p E_p^p = \langle\langle m|\hat{N}|m\rangle\rangle = N,$$

the algebra (3·2) can be regarded as the $su(\mathcal{M})$ algebra by the same reason as the case of the E-$u(\mathcal{M})$ algebra.

which can be easily derived by a rearrangement of variables x^*_{ap} and x_{ap}. Then it is apparent that the Casimir operators are commuting not only with one another but also with all E^p_q and F^a_b, i.e.,

$$\{C^{(k)}_E, E^p_q\} = \{C^{(k)}_E, F^a_b\} = 0, \quad \{C^{(k)}_F, E^p_q\} = \{C^{(k)}_F, F^a_b\} = 0,$$
$$\{C^{(k)}_E, C^{(k')}_F\} = 0. \tag{3·8}$$

We have distinguished the representation carrier space of the F-$u(\mathcal{M})$ algebra as the fictitious space from that of the E-$u(\mathcal{M})$ algebra. By considering the above fact and also that both the $u(\mathcal{M})$ algebras are described by the same variables, this distinction has no meaning. Both the E- and the F-$u(\mathcal{M})$ algebra have the same Casimir operators, and their representation carrier space is common, but only their actions on the representation carrier space are different from each other. Since they are commutable with each other and have common Casimir operators, we call such a couple of algebras the dual algebras, that is, the dual $u(\mathcal{M})$ algebras.[3)]

In what follows, we discuss in more details the duality property in relation to the Casimir operators. We notice the fact that the E-$u(\mathcal{M})$ algebra can be expressed as an $\mathcal{M}$-direct sum of $u(\mathcal{M})$ algebras as follows:

$$E^p_q = \sum_a {}_aE^p_q, \tag{3·9a}$$

$${}_aE^p_q = x^*_{ap}x_{aq}, \quad (a = 1, 2, \cdots, \mathcal{M}) \tag{3·9b}$$

which satisfy the commutation relations

$$\{{}_aE^p_q, {}_bE^r_s\} = \delta_{ab}(\delta_{rq}\ {}_aE^p_s - \delta_{ps}\ {}_aE^r_q), \tag{3·10a}$$

$$\{F^b_c, {}_aE^r_s\} = (\delta_{ac} - \delta_{ab})x^*_{br}x_{cs}. \tag{3·10b}$$

Hence the set $\{{}_aE^p_q; p, q = 1, 2, \cdots, \mathcal{M}\}$ also generates a $u(\mathcal{M})$ algebra. Hereafter we denote this algebra by the symbol ${}_au(\mathcal{M})$ simply.

Then the Casimir operator of k-th order for the ${}_au(\mathcal{M})$ algebra is defined by

$${}_aC^{(k)} = \sum_{p_1, p_2, \cdots, p_k} {}_aE^{p_1}_{p_2}\ {}_aE^{p_2}_{p_3} \cdots {}_aE^{p_k}_{p_1}. \tag{3·11}$$

This is rewritten by a rearrangement of variables in the following form:

$${}_aC^{(k)} = \underbrace{F^a_a F^a_a \cdots F^a_a}_{k \text{ times}}. \tag{3·12}$$

Therefore, only the first-order Casimir operator ${}_aC^{(1)}$ is meaningful:

$${}_aC^{(1)} = F^a_a = n_a, \tag{3·13}$$

which expresses the occupation number of the single-particle level a in the vacuum $|0\rangle\rangle$.

The following relations are easily ascertained to hold:

$$\{{}_aC^{(1)}, {}_bE^p_q\} = 0, \quad \{{}_aC^{(1)}, E^p_q\} = 0, \quad \{{}_aC^{(1)}, C^{(k)}_E\} = 0. \tag{3·14}$$

Therefore, the irreducible representation of the E-$u(\mathcal{M})$ algebra is specified by the set of values associated with the set of the Casimir operators

$$\{({}_aC^{(1)}; a = 1, 2, \cdots, \mathcal{M}),\ (C^{(k)}_E; k = 1, 2, \cdots)\}. \tag{3·15}$$

The above discussion is the conventional description of the $u(\mathcal{M})$ algebra, although the set of Casimir operators ${}_aC^{(1)}$ never appear in the conventional description.

With the use of the dual structure, we can prescribe the irreducible representation carrier space of the E-$u(\mathcal{M})$ algebra in terms of that of the F-$u(\mathcal{M})$ algebra.

As same as the preceding discussion, we express the F-$u(\mathcal{M})$ algebra as an $\mathcal{M}$-direct sum of $u(\mathcal{M})$ algebras as follows:

$$F^a_b = \sum_p {}^pF^a_b, \tag{3·16a}$$

$${}^pF^a_b = x^*_{ap}x_{bp}, \quad (p = 1, 2, \cdots, \mathcal{M}) \tag{3·16b}$$

which obey the relations

$$\{{}^pF^a_b, {}^qF^c_d\} = \delta_{pq}(\delta_{bc}\ {}^pF^a_d - \delta_{ad}\ {}^pF^c_b), \tag{3·17a}$$

$$\{E^p_q, {}^rF^a_b\} = (\delta_{rq} - \delta_{rp})x^*_{ap}x_{bq}. \tag{3·17b}$$

Because of the above relations, the set $\{{}^pF^a_b; a, b = 1, 2, \cdots, \mathcal{M}\}$ generates a $u(\mathcal{M})$ algebra denoted simply by ${}^pu(\mathcal{M})$.

The Casimir operator of the k-th order for the ${}^pu(\mathcal{M})$ algebra is given by

$${}^pC^{(k)} = \sum_{a_1, a_2, \cdots, a_k} {}^pF^{a_1}_{a_2}\ {}^pF^{a_2}_{a_3} \cdots\ {}^pF^{a_k}_{a_1}, \tag{3·18}$$

which can be rearranged in the form

$${}^pC^{(k)} = \underbrace{E^p_pE^p_p \cdots E^p_p}_{k \text{ times}}. \tag{3·19}$$

Therefore, only the first-order Casimir operator ${}^pC^{(1)}$ is independent:

$$\begin{aligned} {}^pC^{(1)} &= E^p_p = \left\langle\left\langle m \middle| \sum_m \hat{c}^\dagger_{pjm}\hat{c}_{pjm} \middle| m \right\rangle\right\rangle \\ &= N_p, \end{aligned} \tag{3·20}$$

which expresses the classical image of the occupation number operator in the single-particle level p. It is easily ascertained that ${}^pC^{(1)}$ satisfies the following commutation relations:

$$\{{}^pC^{(1)}, {}^qF^a_b\} = 0, \quad \{{}^pC^{(1)}, F^a_b\} = 0, \quad \{{}^pC^{(1)}, C^{(k)}_F\} = 0. \tag{3·21}$$

Owing to these relations, the irreducible representation of the F-$u(\mathcal{M})$ algebra can be also specified with the aid of the set of values associated with the set of the Casimir operators

$$\{({}^pC^{(1)}; p = 1, 2, \cdots, \mathcal{M}), (C_F^{(k)}; k = 1, 2, \cdots)\}. \tag{3·22}$$

At the end of this section, we reinvestigate the physical ingredient of the constraint (2·25)

$$\sum_p x_{ap}^* x_{bp} = \delta_{ab} n_a,$$

which is rewritten in the following form:

$$F_b^a = \delta_{ab} n_a. \tag{3·23}$$

This implies that the Casimir operator $C_E^{(k)}$ is reduced to ${}_aC_F^{(1)}$ under the constraints, i.e.,

$$\begin{aligned} C_E^{(k)} &= \sum_{p_1, p_2, \cdots, p_k} E_{p_2}^{p_1} E_{p_3}^{p_2} \cdots E_{p_1}^{p_k} = \sum_{a_1, a_2, \cdots, a_k} F_{a_2}^{a_1} F_{a_3}^{a_2} \cdots F_{a_1}^{a_k} \\ &= \sum_a \{{}_aC_F^{(1)}\}^k \\ &= \sum_a {n_a}^k. \end{aligned} \tag{3·24}$$

This means that the irreducible representation of E-$u(\mathcal{M})$ algebra can be simply prescribed by the set of values $\{n_a;\ a = 1, 2, \cdots, \mathcal{M}\}$. This point will be discussed concretely with the use of $u(2)$ algebra in §6.

§4. Canonical quantization and carrier space

In order to get the irreducible representation carrier space of the $u(\mathcal{M})$ algebra described by $\hat{E}_q^p$, we must prescribe the actions of $\hat{E}_q^p$ on their carrier space. For this aim, we can adopt the Fock-Bargmann representation which just corresponds to, so to speak, an x^* representation:

$$(x_{ap}^*, x_{ap}) \longrightarrow \left(x_{ap}^*, \frac{\partial}{\partial x_{ap}^*}\right). \tag{4·1}$$

In this paper, however, we adopt the method of canonical quantization of boson, because we do not need the explicit form of wave functions in the x^* representation.

4.1. *Requantization*

The requantization can be easily performed by a canonical quantization as follows:

$$(x_{ap}, x_{ap}^*) \longrightarrow (\hat{x}_{ap}, \hat{x}_{ap}^\dagger), \tag{4·2}$$

and their commutation relations are given by

$$[\hat{x}_{ap}, \hat{x}_{bq}^\dagger]_- = \delta_{ab}\delta_{pq}, \quad [\hat{x}_{ap}, \hat{x}_{bq}]_- = [\hat{x}_{ap}^\dagger, \hat{x}_{bq}^\dagger]_- = 0. \tag{4·3}$$

Then the classical images are requantized as

$$\hat{E}^p_q = \sum_a \hat{x}^\dagger_{ap}\hat{x}_{aq}, \quad \hat{F}^a_b = \sum_p \hat{x}^\dagger_{ap}\hat{x}_{bp}. \tag{4·4}$$

The algebra defined by the use of the Poisson bracket is requantized by

$$\begin{aligned}[\hat{E}^p_q, \hat{E}^r_s]_- &= \delta_{qr}\hat{E}^p_s - \delta_{ps}\hat{E}^r_q,\\ [\hat{F}^a_b, \hat{F}^c_d]_- &= \delta_{bc}\hat{F}^a_d - \delta_{ad}\hat{F}^c_b,\\ [\hat{E}^p_q, \hat{F}^a_b]_- &= 0.\end{aligned} \tag{4·5}$$

Therefore, the requantized operators $\hat{E}^p_q$ and $\hat{F}^a_b$ reproduce respectively the E- and the F-$u(\mathcal{M})$ algebra.

The Casimir operators for the E- and the F-$u(\mathcal{M})$ algebra are given in a requantized version by

$$\hat{C}^{(k)}_E = \sum_{p_1,p_2,\cdots,p_k} \hat{E}^{p_1}_{p_2}\hat{E}^{p_2}_{p_3}\cdots\hat{E}^{p_k}_{p_1}, \tag{4·6}$$

$$\hat{C}^{(k)}_F = \sum_{a_1.a_2,\cdots,a_k} \hat{F}^{a_1}_{a_2}\hat{F}^{a_2}_{a_3}\cdots\hat{F}^{a_k}_{a_1}. \tag{4·7}$$

Even in the requantized version, the two types of Casimir operators are identical to each other

$$\hat{C}^{(k)}_E = \hat{C}^{(k)}_F. \tag{4·8}$$

This is ascertained by a rearrangement of variables, although the variables are not always mutually commutable in this case. Then it is apparent that the Casimir operators are commutable not only with each other but also with all $\hat{E}^p_q$ and $\hat{F}^a_b$, i.e.,

$$[\hat{C}^{(k)}_E, \hat{E}^p_q]_- = [\hat{C}^{(k)}_E, \hat{F}^a_b]_- = 0, \quad [\hat{C}^{(k)}_E, \hat{C}^{(k')}_F]_- = 0. \tag{4·9}$$

In this way, the duality of the couple of the E- and the F-$u(\mathcal{M})$ algebra is preserved even in the requantized version.

4.2. *Orthonormal basis of representation carrier space*

Here we discuss how to specify the irreducible representation of the E-$u(\mathcal{M})$ algebra in the requantized version. The E-$u(\mathcal{M})$ algebra can be expressed as an $\mathcal{M}$-direct sum of $u(\mathcal{M})$ algebras, i.e., E-$u(\mathcal{M}) = \bigoplus_a {}_a u(\mathcal{M})$:

$$\hat{E}^p_q = \sum_a {}_a\hat{E}^p_q, \tag{4·10a}$$

$${}_a\hat{E}^p_q = \hat{x}^\dagger_{ap}\hat{x}_{aq}, \quad (a = 1, 2, \cdots, \mathcal{M}) \tag{4·10b}$$

and the following commutation relations are satisfied:

$$[{}_a\hat{E}^p_q, {}_b\hat{E}^r_s]_- = \delta_{ab}(\delta_{qr}\ {}_a\hat{E}^p_s - \delta_{ps}\ {}_a\hat{E}^r_q), \tag{4·11a}$$

$$[\hat{F}^b_c, {}_a\hat{E}^r_s]_- = (\delta_{ac} - \delta_{ab})\hat{x}^\dagger_{br}\hat{x}_{cs}. \tag{4·11b}$$

The Casimir operator of the k-th order of the ${}_a u(\mathcal{M})$ algebra is given by

$$ {}_a\hat{C}^{(k)} = \sum_{p_1,p_2,\cdots,p_k} {}_a\hat{E}^{p_1}_{p_2}\, {}_a\hat{E}^{p_2}_{p_3} \cdots {}_a\hat{E}^{p_k}_{p_1}. \tag{4·12} $$

By rearranging the variables, we can get the expression

$$ {}_a\hat{C}^{(k)} = f^{(k)}(\hat{F}^a_a), \tag{4·13} $$

where $f^{(k)}$ is a polynomial of order k. For instance,

$$ \begin{aligned} {}_a\hat{C}^{(1)} &= \sum_p \hat{x}^\dagger_{ap}\hat{x}_{ap} = \hat{F}^a_a \\ &= \hat{n}_a, \end{aligned} \tag{4·14a} $$

$$ {}_a\hat{C}^{(2)} = (\hat{F}^a_a)^2 + (\mathcal{M}-1)\hat{F}^a_a. \tag{4·14b} $$

Thus the independent Casimir operator is only the first-order one, ${}_a\hat{C}^{(1)}$, which satisfies

$$ [{}_a\hat{C}^{(1)}, {}_b\hat{E}^p_q]_- = [{}_a\hat{C}^{(1)}, \hat{E}^p_q]_- = 0, \quad [{}_a\hat{C}^{(1)}, \hat{C}^{(k)}_E]_- = 0. \tag{4·15} $$

Therefore the basis vector of the irreducible representation carrier space of the E-$u(\mathcal{M})$ algebra is specified by a set of eigenvalues of the Casimir operators

$$ \{({}_a\hat{C}^{(1)}; a = 1, 2, \cdots, \mathcal{M}), (\hat{C}^{(k)}_E; k = 1, 2\cdots)\}. \tag{4·16} $$

In order to get the orthonormal basis of the irreducible representation carrier space, we first introduce eigenstates of $\{{}_a\hat{C}^{(1)}; a = 1, 2, \cdots, \mathcal{M}\}$. We define clusters as

$$ [\hat{x}^\dagger_a]^{[\sum_p n_{ap}=n_a]} = \prod_p \frac{1}{\sqrt{n_{ap}!}}(\hat{x}^\dagger_{ap})^{n_{ap}}. \quad \left(\sum_p n_{ap} = n_a, a = 1, 2, \cdots, \mathcal{M}\right) \tag{4·17} $$

Then the cluster state is defined by

$$ \left|\left[\sum_p n_{ap} = n_a\right]\right\rangle = [\hat{x}^\dagger_a]^{[\sum_p n_{ap}=n_a]}|0\rangle_x, \tag{4·18} $$

where $|0\rangle_x$ is the vacuum of the boson field $\hat{x}^\dagger_{ap}$, i.e.,

$$ \hat{x}_{ap}|0\rangle_x = 0. \tag{4·19} $$

This cluster state satisfies the relations

$$ \left\langle\left[\sum_p n_{ap} = n_a\right]\middle|\left[\sum_p n_{bp} = n_b\right]\right\rangle = \delta_{ab}\prod_p \delta_{n_{ap}n_{bp}}, \tag{4·20a} $$

$$ \begin{aligned} &{}_b\hat{E}^p_q\left|\left[\sum_r n_{ar} = n_a\right]\right\rangle \\ &= \begin{cases} \delta_{ab} n_{ap}|[\sum_r n_{ar} = n_a]\rangle\,; & p = q, \\ \delta_{ab}\sqrt{(n_{ap}+1)n_{aq}}|[\sum_{r(\neq p,q)} n_{ar} + (n_{ap}+1) + (n_{aq}-1) = n_a]\rangle\,; & p \neq q, \end{cases} \end{aligned} \tag{4·20b} $$

$$ {}_b\hat{C}^{(1)}\Big|\Big[\sum_p n_{ap}=n_a\Big]\Big\rangle=\delta_{ab}n_a\Big|\Big[\sum_p n_{ap}=n_a\Big]\Big\rangle. \tag{4·20c} $$

The cluster states $|[\sum_p n_{ap}=n_a]\rangle$ with a fixed n_a span an invariant subspace of ${}_au(\mathcal{M})$ algebra.

Direct products of the cluster states provide the orthonormal basis of representation carrier space:

$$ \Big|\prod_a\Big[\sum_p n_{ap}=n_a\Big]:n\Big\rangle=\prod_a[\hat{x}_a^\dagger]^{[\sum_p n_{ap}=n_a]}|0\rangle_x. \quad \Big(n=\sum_a n_a\Big) \tag{4·21} $$

Then the action of $\hat{E}_q^p$ on this space is given as follows:

$$ \begin{aligned} \hat{E}_p^p\Big|\prod_a\Big[\sum_p n_{ap}=n_a\Big]:n\Big\rangle &= \hat{N}_p\Big|\prod_a\Big[\sum_p n_{ap}=n_a\Big]:n\Big\rangle \\ &= \sum_b n_{bp}\Big|\prod_a\Big[\sum_p n_{ap}=n_a\Big]:n\Big\rangle, \end{aligned} \tag{4·22a} $$

$$ \begin{aligned} &\hat{E}_q^p\Big|\prod_a\Big[\sum_p n_{ap}=n_a\Big]:n\Big\rangle \\ &\quad=\sum_b\sqrt{(n_{bp}+1)n_{bq}}\Big|\Big[\sum_{r(\neq p,q)} n_{br}+(n_{bp}+1)+(n_{bq}-1)=n_b\Big] \\ &\qquad\qquad\times\prod_{a(\neq b)}\Big[\sum_r n_{ar}=n_a\Big]:n\Big\rangle. \quad (p\neq q) \end{aligned} \tag{4·22b} $$

From Eq. (4·22), the nucleon number N_p of the single-particle level p and the total nucleon number in the state $|\prod_a[\sum_p n_{ap}=n_a]:n\rangle$ are respectively given by

$$ N_p=\sum_a n_{ap}, \quad N=\sum_p N_p=\sum_p\sum_a n_{ap}=\sum_a n_a=n. \tag{4·23} $$

As obvious from the commutation relation $[\hat{F}_a^a,\hat{E}_q^p]_-=0$, the quantity n_a as the eigenvalue of $\hat{F}_a^a$ is a good quantum number. This fact will play an essential role in the discussions of the next section. Then the actions of Casimir operators $\hat{C}_E^{(1)}$ and $\hat{C}_E^{(2)}$ are given as follows:

$$ \begin{aligned} &\hat{C}_E^{(1)}\Big|\prod_a\Big[\sum_p n_{ap}=n_a\Big]:n\Big\rangle \\ &\quad=\hat{N}\Big|\prod_a\Big[\sum_p n_{ap}=n_a\Big]:n\Big\rangle \\ &\quad=n\Big|\prod_a\Big[\sum_p n_{ap}=n_a\Big]:n\Big\rangle, \end{aligned} \tag{4·24a} $$

$$
\begin{aligned}
&\hat{C}_E^{(2)}\Big|\prod_a\Big[\sum_p n_{ap}=n_a\Big]:n\Big\rangle \\
&\quad=\sum_p\Big(\sum_b n_{bp}\Big)^2\Big|\prod_a\Big[\sum_r n_{ar}=n_a\Big]:n\Big\rangle \\
&\qquad+\sum_{p\neq q}\Big\{\sum_b(n_{bp}+1)n_{bq}\Big|\prod_a\Big[\sum_r n_{ar}=n_a\Big]:n\Big\rangle \\
&\qquad\quad+\sum_{b\neq c}\sqrt{(n_{bp}+1)n_{cp}(n_{cq}+1)n_{bq}} \\
&\qquad\quad\times\Big|\Big[\sum_{r(\neq p,q)} n_{br}+(n_{bp}+1)+(n_{bq}-1)=n_b\Big] \\
&\qquad\qquad\times\Big[\sum_{r(\neq p,q)} n_{cr}+(n_{cp}-1)+(n_{cq}+1)=n_c\Big] \\
&\qquad\qquad\times\prod_{a(\neq b.c)}\Big[\sum_r n_{ar}=n_a\Big]:n\Big\rangle\Big\}. \qquad (4\cdot24\text{b})
\end{aligned}
$$

Equation (4·24) shows that, in the irreducible representation, the total nucleon number N is always a good quantum number. We can get an irreducible representation by diagonalizing the Casimir operators $\{\hat{C}_E^{(k)}; k=1,2,\cdots\}$ with respect to the present basis vectors.

Here we go back to the set of variables $\{(X_{ap}^*, X_{ap}, (n_a,\phi_a); a,p=1,2,\cdots,\mathcal{M}\}$. The canonical transformation (2·21) is given in its requantized version as

$$(\hat{x}_{ap},\hat{x}_{ap}^\dagger)=(\exp[-i\hat{\phi}_a]\hat{X}_{ap},\hat{X}_{ap}^\dagger\exp[i\hat{\phi}_a]);\quad a\neq p, \qquad (4\cdot25\text{a})$$

$$(\hat{x}_{aa},\hat{x}_{aa}^\dagger)=\left(\exp[-i\hat{\phi}_a]\sqrt{\hat{n}_a-\sum_{p(\neq a)}\hat{X}_{ap}^\dagger\hat{X}_{ap}},\sqrt{\hat{n}_a-\sum_{p(\neq a)}\hat{X}_{ap}^\dagger\hat{X}_{ap}}\exp[i\hat{\phi}_a]\right). \qquad (4\cdot25\text{b})$$

Then we have the commutation relations

$$[\hat{X}_{ap},\hat{X}_{bq}^\dagger]_-=\delta_{ab}\delta_{pq},\quad [\hat{X}_{ap},\hat{X}_{bq}]_-=0, \qquad (4\cdot26\text{a})$$

$$[\hat{\phi}_a,\hat{n}_b]_-=i\delta_{ab},\quad [\hat{\phi}_a,\hat{\phi}_b]_-=[\hat{n}_a,\hat{n}_b]_-=0, \qquad (4\cdot26\text{b})$$

$$[\hat{X}_{ap},\hat{\phi}_b]_-=[\hat{X}_{ap},\hat{n}_b]_-=0. \qquad (4\cdot26\text{c})$$

The generators $\hat{E}_q^p$ are expressed in the following form:

$$
\begin{aligned}
\hat{E}_q^p&=\sum_a\hat{x}_{ap}^\dagger\hat{x}_{aq} \\
&=\begin{cases}
\sum_{a(\neq p,q)}\hat{X}_{ap}^\dagger\hat{X}_{aq}+\sqrt{\hat{n}_p-\sum_{r(\neq p)}\hat{X}_{pr}^\dagger\hat{X}_{pr}}\;\hat{X}_{pq} & \\
\qquad+\hat{X}_{qp}^\dagger\sqrt{\hat{n}_q-\sum_{r(\neq q)}\hat{X}_{qr}^\dagger\hat{X}_{qr}}\,; & p\neq q, \\
\sum_{a(\neq p)}\hat{X}_{ap}^\dagger\hat{X}_{ap}+(\hat{n}_p-\sum_{r(\neq p)}\hat{X}_{pr}^\dagger\hat{X}_{pr})\,; & p=q,
\end{cases}
\end{aligned}
\qquad (4\cdot27)
$$

and the cluster $[\sum_p n_{ap} = n_a]$ is written by

$$\left[\sum_p n_{ap} = n_a\right] = \exp[in_a\hat{\phi}_a] \prod_{p(\neq a)} \frac{1}{\sqrt{n_{ap}!}} (\hat{X}^\dagger_{ap})^{n_{ap}}. \tag{4·28}$$

At the end of this section, we discuss the basis of the representation carrier space with the use of the F-$u(\mathcal{M})$ algebra. This algebra is also expressed as an $\mathcal{M}$-direct sum of $u(\mathcal{M})$ algebras, $\bigoplus_p {}^p u(\mathcal{M})$:

$$\hat{F}^a_b = \sum_p {}^p\hat{F}^a_b, \tag{4·29a}$$

$${}^p\hat{F}^a_b = \hat{x}^\dagger_{ap}\hat{x}_{bp}, \quad (p = 1, 2, \cdots, \mathcal{M}) \tag{4·29b}$$

which satisfy the commutation relations

$$[{}^p\hat{F}^a_b, {}^q\hat{F}^c_d]_- = \delta_{pq}(\delta_{cb}\, {}^p\hat{F}^a_d - \delta_{ad}\, {}^p\hat{F}^c_b), \tag{4·30a}$$

$$[\hat{E}^p_q, {}^r\hat{F}^a_b]_- = (\delta_{rq} - \delta_{rp})\hat{x}^\dagger_{ap}\hat{x}_{bq}. \tag{4·30b}$$

The first of the above relations shows that the set $\{{}^p\hat{F}^a_b; a, b = 1, 2, \cdots, \mathcal{M}\}$ generates a $u(\mathcal{M})$ algebra denoted by ${}^p u(\mathcal{M})$. The Casimir operator of the k-th order is given by

$${}^p\hat{C}^{(k)} = \sum_{a_1, a_2, \cdots, a_k} {}^p\hat{F}^{a_1}_{a_2}\, {}^p\hat{F}^{a_2}_{a_3} \cdots {}^p\hat{F}^{a_k}_{a_1}. \tag{4·31}$$

The above operator can be rewritten by rearranging variables in the form

$${}^p\hat{C}^{(k)} = g^{(k)}(\hat{E}^p_p), \tag{4·32}$$

where $g^{(k)}$ is a polynomial of order k. For instance,

$${}^p\hat{C}^{(2)} = (\hat{E}^p_p)^2 + (\mathcal{M} - 1)\hat{E}^p_p. \tag{4·33}$$

In this case too, therefore, only the 1-st order Casimir operator is meaningful:

$${}^p\hat{C}^{(1)} = \hat{E}^p_p = \hat{N}_p, \tag{4·34}$$

which expresses the nucleon number operator of the single-particle level p. Then we have the following commutation relations:

$$[{}^p\hat{C}^{(1)}, {}^q\hat{F}^a_b]_- = [{}^p\hat{C}^{(1)}, \hat{F}^a_b]_- = [{}^p\hat{C}^{(1)}, \hat{C}^{(k)}_F]_- = 0. \tag{4·35}$$

Therefore the irreducible representation can be specified in terms of the eigenvalues of the Casimir operators

$$\{({}^p\hat{C}^{(1)}; p = 1, 2, \cdots, \mathcal{M}), (\hat{C}^{(k)}_F; k = 1, 2, \cdots)\}. \tag{4·36}$$

The orthonormal basis of the representation carrier space is constructed by the following way: We introduce clusters

$$[\hat{x}^\dagger_p]^{[\sum_a N_{ap} = N_p]} = \prod_a \frac{1}{\sqrt{N_{ap}!}} (\hat{x}^\dagger_{ap})^{N_{ap}}. \quad \left(\sum_a N_{ap} = N_p, p = 1, 2, \cdots, \mathcal{M}\right) \tag{4·37}$$

Then the orthonormal basis vector is written as

$$\Big|\prod_p\Big[\sum_a N_{ap}=N_p\Big]:N\Big\rangle=\prod_p[\hat{x}_p^\dagger]^{[\sum_a N_{ap}=N_p]}|0\rangle_x. \quad \Big(\sum_p N_p=N\Big) \tag{4·38}$$

The above basis vector satisfies the relations:

$$\Big\langle\prod_p\Big[\sum_a N'_{ap}=N'_p\Big]:N'\Big|\prod_p\Big[\sum_a N_{ap}=N_p\Big]:N\Big\rangle=\delta_{NN'}\prod_{a,p}\delta_{N_{ap}N'_{ap}}, \tag{4·39a}$$

$$^p\hat{C}^{(1)}\Big|\prod_r\Big[\sum_a N_{ar}=N_r\Big]:N\Big\rangle=N_p\Big|\prod_r\Big[\sum_a N_{ar}=N_r\Big]:N\Big\rangle, \tag{4·39b}$$

$$\hat{C}_E^{(1)}\Big|\prod_r\Big[\sum_a N_{ar}=N_r\Big]:N\Big\rangle=N\Big|\prod_r\Big[\sum_a N_{ar}=N_r\Big]:N\Big\rangle, \tag{4·39c}$$

$$\begin{aligned}&\hat{E}_q^p\Big|\prod_r\Big[\sum_a N_{ar}=N_r\Big]:N\Big\rangle\\&=\sum_c\sqrt{(N_{cp}+1)N_{cq}}\Big|\Big[\sum_{a(\neq c)}N_{ap}+(N_{cp}+1)=N_p+1\Big]\\&\quad\times\Big[\sum_{a(\neq c)}N_{aq}+(N_{cq}-1)=N_q-1\Big]\prod_{r(\neq p,q)}\Big[\sum_a N_{ar}=N_r\Big]:N\Big\rangle, \quad (p\neq q)\end{aligned} \tag{4·39d}$$

$$\begin{aligned}&\hat{C}_E^{(2)}\Big|\prod_r\Big[\sum_a N_{ar}=N_r\Big]:N\Big\rangle\\&=\Big\{\sum_p {N_p}^2+\sum_{p\neq q}\sum_c(N_{cp}+1)N_{cq}\Big\}\Big|\prod_r\Big[\sum_a N_{ar}=N_r\Big]:N\Big\rangle\\&\quad+\sum_{p\neq q}\sum_{c\neq d}\sqrt{(N_{cp}+1)N_{dp}(N_{cq}+1)N_{cq}}\\&\qquad\times\Big|\Big[\sum_{a(\neq c,d)}N_{ap}+(N_{cp}+1)+(N_{dp}-1)=N_p\Big]\\&\qquad\times\Big[\sum_{a(\neq c,d)}N_{aq}+(N_{cq}-1)+(N_{dq}+1)=N_q\Big]\prod_{r(\neq p,q)}\Big[\sum_a N_{ar}=N_r\Big]:N\Big\rangle.\end{aligned} \tag{4·39e}$$

The set of the above state vectors provides an orthonormal basis of the representation carrier space. However, these vectors do not belong to the eigenvalues of the Casimir operators $\{\hat{C}_E^{(k)};k=2,\cdots\}$. In this sense, these vectors express certain mixed states, that is, the space spanned by these vectors is reducible. The irreducible vector space can be obtained by diagonalizing the Casimir operators with respect to this basis.

The above two types of basis span the very same vector space. In fact, the following relation holds:

$$\Big\langle\prod_a\Big[\sum_p n_{ap}=n_a\Big]:n\Big|\prod_p\Big[\sum_a N_{ap}=N_p\Big]:N\Big\rangle=\delta_{nN}\prod_{a,p}\delta_{n_{ap}N_{ap}}. \tag{4·40}$$

From this relation we have the following duality in the occupancy n_{ap}:

$$n_a = \sum_p n_{ap}, \quad N_p = \sum_a N_{ap},$$

$$n = \sum_a n_a = \sum_a \sum_p n_{ap} = \sum_p \sum_a N_{ap} = \sum_p N_p = N. \tag{4·41}$$

§5. Retractable case

In this section, we take up two cases which are useful for the schematic description of nuclear collective motions, where single-particle levels are vacant except for the 1-st and the 2-nd level in the vacuum of the mixed state. That is to say, the occupation number n_a of the single-particle level a in $|0\rangle\rangle$ vanishes for $a \neq 1, 2$:

$$n_a = 0. \quad (a = 3, 4, \cdots, \mathcal{M}) \tag{5·1}$$

From the definitions (2·15) and (2·21), we have

$$x_{ap} = x^*_{ap} = 0. \quad (a = 3, 4, \cdots, \mathcal{M}) \tag{5·2}$$

This situation gives rise to a significant effect on the structure of E^p_q and F^a_b:

$$E^p_q = \sum_{a=1}^{\mathcal{M}} x^*_{ap} x_{aq} \longrightarrow E^p_q = \sum_{a=1}^{2} x^*_{ap} x_{aq}, \tag{5·3a}$$

$$F^a_b = \sum_{p=1}^{\mathcal{M}} x^*_{ap} x_{bp} \longrightarrow F^a_b = \begin{cases} \sum_{p=1}^{\mathcal{M}} x^*_{ap} x_{bp}\,; & a, b = 1, 2, \\ 0 \qquad\qquad ; & \text{otherwise.} \end{cases} \tag{5·3b}$$

The retraction (5·3a) does not affect the algebraic structure of the E-$u(\mathcal{M})$ algebra. The retraction (5·3b), however, seriously affects that of the F-$u(\mathcal{M})$ algebra and brings about the reduction of the $u(\mathcal{M})$ algebra to a $u(1)$ or a $u(2)$ algebra.

We can show in the requantized version that the above reduction persists. As has been shown by the relations (4·22) and (4·24), the set $\{n_a; a = 1, 2, \cdots, \mathcal{M}\}$ is fixed in an irreducible representation carrier space. Conversely, a choice of the set $\{n_a = 0; a = 3, 4, \cdots, \mathcal{M}\}$ restricts the representation carrier space to its subspace as

$$\left\{ \left| \prod_{a=1}^{\mathcal{M}} \left[\sum_p n_{ap} = n_a \right] : n \right\rangle \right\} \longrightarrow \left\{ \left| \prod_{a=1}^{2} \left[\sum_p n_{ap} = n_a \right] : n \right\rangle \right\}. \tag{5·4}$$

In association with the restriction of the representation carrier space, we are able to retract the generators in the following way:

$$\hat{E}^p_q = \sum_{a=1}^{\mathcal{M}} \hat{x}^\dagger_{ap} \hat{x}_{aq} \longrightarrow \hat{E}^p_q = \sum_{a=1}^{2} \hat{x}^\dagger_{ap} \hat{x}_{aq}. \tag{5·5}$$

Of course, this retraction preserves the algebraic structure of the E-$u(\mathcal{M})$, i.e.,

$$[\hat{E}^p_q, \hat{E}^r_s]_- = \delta_{rq} \hat{E}^p_s - \delta_{ps} \hat{E}^r_q.$$

Contrary to the E-$u(\mathcal{M})$ algebra, the restriction of the representation carrier space affects seriously the F-$u(\mathcal{M})$ algebra:

$$\hat{F}^a_b = \sum_{p=1}^{\mathcal{M}} \hat{x}^\dagger_{ap}\hat{x}_{bp} \longrightarrow \hat{F}^a_b = \begin{cases} \sum_{p=1}^{\mathcal{M}} \hat{x}^\dagger_{ap}\hat{x}_{bp}\,; & a,b=1,2, \\ 0 \qquad\qquad ; & \text{otherwise.} \end{cases} \tag{5·6}$$

That is to say,

$$F\text{-}u(\mathcal{M}) \longrightarrow F\text{-}u(1)\{n_1 \neq 0, n_a = 0\ (a=2,3,\cdots,\mathcal{M})\}, \tag{5·7}$$

$$F\text{-}u(\mathcal{M}) \longrightarrow F\text{-}u(2)\{n_1, n_2 \neq 0, n_a = 0\ (a=3,4,\cdots,\mathcal{M})\}. \tag{5·8}$$

Therefore, it is a straightforward task to obtain the irreducible representation of the retractable F-$u(\mathcal{M})$ algebra, i.e., the retracted F-$u(1)$ or the retracted F-$u(2)$ algebra. Due to the duality, the irreducible representation thus obtained is nothing but that of the E-$u(\mathcal{M})$ algebra, the basis of the representation carrier space of which is just given by

$$\left\{\left|\prod_{a=1}^{2}\left[\sum_p n_{ap} = n_a\right] : n\right\rangle\right\}. \tag{5·9}$$

In what follows, we investigate how to retain the duality under the algebraic retraction discussed above.

5.1. *Retracted F-u(1) algebra*

First we take up the case $n_a = 0$ $(a = 2, 3, \cdots, \mathcal{M})$, where n_a denotes the nucleon number occupying the single-particle level a in the vacuum $|0\rangle\rangle$. Following the above discussion, generators can be reduced to as follows:

$$\hat{E}^p_q = \hat{x}^\dagger_{1p}\hat{x}_{1q}, \tag{5·10a}$$

$$\hat{F}^a_b = \delta_{a1}\delta_{1b}\hat{F}^1_1, \quad \hat{F}^1_1 = \sum_{p=1}^{\mathcal{M}} \hat{x}^\dagger_{1p}\hat{x}_{1p}. \tag{5·10b}$$

They satisfy the algebra

$$[\hat{E}^p_q, \hat{E}^r_s]_- = \delta_{rq}\hat{E}^p_s - \delta_{ps}\hat{E}^r_q, \tag{5·11a}$$

$$[\hat{F}^a_b, \hat{F}^c_d]_- = 0, \tag{5·11b}$$

$$[\hat{E}^p_q, \hat{F}^a_b]_- = 0. \tag{5·11c}$$

The set of $\hat{E}^p_q$ still obeys the $u(\mathcal{M})$ algebra. On the other hand, the set of $\hat{F}^a_b$ is retracted to the $u(1)$ algebra. Hence its Casimir operator is just $\hat{F}^1_1$ itself.

The Casimir operators for the E-$u(\mathcal{M})$ algebra can be written as

$$\begin{aligned} \hat{C}^{(k)}_E &= [{}_a\hat{C}^{(k)}]_{a=1} \\ &= f^{(k)}(\hat{F}^1_1). \end{aligned} \tag{5·12}$$

Therefore, in this case, the irreducible representation of the E-$u(\mathcal{M})$ algebra can be specified with the value determined by the Casimir operator $\hat{F}_1^1$, the eigenvalue of which is denoted by n_1.

The orthonormal basis of the representation carrier space is given by

$$\left|\left[\sum_{p=1}^{\mathcal{M}} n_{1p} = n_1\right] : n_1\right\rangle, \text{ i.e., } \left|\prod_{p=1}^{\mathcal{M}} [n_{1p}] : n_1\right\rangle. \tag{5·13}$$

That is,

$$\left|\left[\sum_{p=1}^{\mathcal{M}} n_{1p} = n_1\right] : n_1\right\rangle = \prod_{p=1}^{\mathcal{M}} \frac{1}{\sqrt{n_{1p}!}} (\hat{x}_{1p}^\dagger)^{n_{1p}} |0\rangle_x, \quad \left(\sum_{p=1}^{\mathcal{M}} n_{1p} = n_1\right) \tag{5·14}$$

which satisfies

$$\hat{F}_1^1 \left|\left[\sum_{p=1}^{\mathcal{M}} n_{1p} = n_1\right] : n_1\right\rangle = n_1 \left|\left[\sum_{p=1}^{\mathcal{M}} n_{1p} = n_1\right] : n_1\right\rangle. \tag{5·15}$$

Therefore, the space spanned by the basis vectors given above is an irreducible representation carrier space belonging to the eigenvalue n_1 of the Casimir operator $\hat{F}_1^1$. The value n_1 is just specified by the total nucleon number.

The action of $\hat{E}_q^p$ on this vector space is given by

$$\hat{E}_p^p \left|\prod_{p=1}^{\mathcal{M}} [n_{1p}] : n_1\right\rangle = n_{1p} \left|\prod_{p=1}^{\mathcal{M}} [n_{1p}] : n_1\right\rangle, \tag{5·16a}$$

$$\hat{E}_q^p \left|\prod_{p=1}^{\mathcal{M}} [n_{1p}] : n_1\right\rangle = \sqrt{(n_{1p}+1)n_{1q}} \left|[n_{1p}+1][n_{1q}-1] \prod_{r(\neq p,q)} [n_{1r}] : n_1\right\rangle; \; p \neq q. \tag{5·16b}$$

In this case, we can obtain a boson representation for $\hat{E}_q^p$ by the following ways: Following the canonical transformation (2·21), we introduce a set of canonical variables

$$\{(\hat{X}_{1p}^\dagger, \hat{X}_{1p}; p = 2, 3, \cdots, \mathcal{M}), (\hat{\phi}, \hat{N})\}, \tag{5·17}$$

which are defined by

$$\hat{x}_{1p} = \exp[-i\hat{\phi}]\hat{X}_{1p}, \quad \hat{x}_{1p}^\dagger = \hat{X}_{1p}^\dagger \exp[i\hat{\phi}], \quad (p = 2, 3, \cdots, \mathcal{M}) \tag{5·18a}$$

$$\hat{x}_{11} = \exp[-i\hat{\phi}]\sqrt{\hat{N} - \sum_{p=2}^{\mathcal{M}} \hat{X}_{1p}^\dagger \hat{X}_{1p}}, \quad \hat{x}_{11}^\dagger = \sqrt{\hat{N} - \sum_{p=2}^{\mathcal{M}} \hat{X}_{1p}^\dagger \hat{X}_{1p}} \; \exp[i\hat{\phi}], \tag{5·18b}$$

and satisfy the commutation relations

$$[\hat{X}_{1p}, \hat{X}_{1q}^\dagger]_- = \delta_{pq}, \quad [\hat{X}_{1p}, \hat{X}_{1q}]_- = [\hat{X}_{1p}^\dagger, \hat{X}_{1q}^\dagger]_- = 0, \tag{5·19a}$$

$$[\hat{\phi}, \hat{N}]_- = i, \tag{5·19b}$$

$$[\hat{X}_{1p}, \hat{\phi}]_- = [\hat{X}_{1p}, \hat{N}]_- = 0. \tag{5·19c}$$

With the use of the new set of variables, we have the following expression:

$$\hat{E}^1_1 = \hat{N} - \sum_{p=2}^{\mathcal{M}} \hat{X}^\dagger_{1p}\hat{X}_{1p},$$

$$\hat{E}^1_p = \sqrt{\hat{N} - \sum_{r=2}^{\mathcal{M}} \hat{X}^\dagger_{1r}\hat{X}_{1r}}\ \hat{X}_{1p}, \quad \hat{E}^p_1 = \hat{X}^\dagger_{1p}\sqrt{\hat{N} - \sum_{r=2}^{\mathcal{M}} \hat{X}^\dagger_{1r}\hat{X}_{1r}},$$

$$\hat{E}^p_q = \hat{X}^\dagger_{1p}\hat{X}_{1q}. \quad (p, q = 2, 3, \cdots, \mathcal{M}) \tag{5·20}$$

The above expression is just the Holstein-Primakoff boson representation.[6)] Our treatment does not premise the number N equal to $2j + 1$, contrary to the conventional treatment.

For this expression, the state vector is rewritten in the form

$$\Big| \prod_{p=2}^{\mathcal{M}} [n_{1p}] : n_1 \Big\rangle = \exp[in_1\hat{\phi}] \prod_{p=2}^{\mathcal{M}} \frac{1}{\sqrt{n_{1p}!}} (\hat{X}^\dagger_{1p})^{n_{1p}} |0\rangle_X. \quad \Big(\sum_{p=2}^{\mathcal{M}} n_{1p} = n_1 \Big) \tag{5·21}$$

5.2. *Retracted F-u(2) algebra*

Next we discuss the second case, i.e.,

$$n_1, n_2 \neq 0, \quad n_a = 0 \text{ for } a = 3, 4, \cdots, \mathcal{M}. \tag{5·22}$$

In this case, we can deal with the E- and the F-$u(\mathcal{M})$ algebra within the subspace

$$\Big\{ \Big| \prod_{a=1}^{2} \Big[\sum_{p=1}^{\mathcal{M}} n_{ap} = n_a \Big] : n \Big\rangle \quad ; n_1 + n_2 = n \Big\}. \tag{5·23}$$

On this subspace, the algebra are retracted in the following form:

$$\hat{E}^p_q = \sum_{a=1}^{2} \hat{x}^\dagger_{ap}\hat{x}_{ap}, \tag{5·24a}$$

$$\hat{F}^a_b = \begin{cases} \sum_{p=1}^{\mathcal{M}} \hat{x}^\dagger_{ap}\hat{x}_{bp}\,; & a, b = 1, 2, \\ 0 \quad ; & \text{otherwise.} \end{cases} \tag{5·24b}$$

In what follows, we use the notations

$$\hat{T}_+ = \hat{F}^1_2, \quad \hat{T}_- = \hat{F}^2_1, \quad \hat{T}_0 = \frac{1}{2}(\hat{F}^1_1 - \hat{F}^2_2),$$

$$\hat{N} = \hat{F}^1_1 + \hat{F}^2_2. \tag{5·25}$$

The present algebra is given by

$$[\hat{E}^p_q, \hat{E}^r_s]_- = \delta_{rq}\hat{E}^p_s - \delta_{ps}\hat{E}^r_q, \tag{5·26a}$$

$$[\hat{T}_0, \hat{T}_\pm]_- = \pm\hat{T}_\pm, \quad [\hat{T}_+, \hat{T}_-]_- = 2\hat{T}_0, \quad [\hat{N}, \hat{T}_j]_- = 0, \tag{5·26b}$$

$$[\hat{E}^p_q, \hat{T}_j]_- = [\hat{E}^p_q, \hat{N}]_- = 0\,. \quad (j = 0, \pm) \tag{5·26c}$$

Therefore, the set of $\hat{E}^p_q$ obeys still the $u(\mathcal{M})$ algebra. On the other hand, the algebra of $\hat{F}^a_b$ is retracted to the $u(2)$ algebra. In this case, we can express the irreducible representation of the E-$u(\mathcal{M})$ algebra in terms of only the $u(2)$ algebra.

The Casimir operator of the k-th order, $\hat{C}^{(k)}_E$, can be given in terms of the Casimir operators $\hat{C}^{(1)}_F$ and $\hat{C}^{(2)}_F$:

$$\hat{C}^{(k)}_E = h^{(k)}(\hat{C}^{(1)}_F, \hat{C}^{(2)}_F),$$

where $h^{(k)}$ is a certain polynomial. For instance,

$$\begin{aligned}
\hat{C}^{(1)}_E &= \hat{C}^{(1)}_F, \\
\hat{C}^{(2)}_E &= \hat{C}^{(2)}_F + (\mathcal{M}-2)\hat{C}^{(1)}_F, \\
\hat{C}^{(3)}_E &= \hat{C}^{(3)}_F + 2(\mathcal{M}-2)\hat{C}^{(2)}_F + (\mathcal{M}-2)^2\hat{C}^{(1)}_F \\
&= \frac{3}{2}\hat{C}^{(1)}_F\hat{C}^{(2)}_F + (2\mathcal{M}-3)\hat{C}^{(2)}_F - \frac{1}{2}(\hat{C}^{(1)}_F)^3 - \frac{1}{2}(\hat{C}^{(1)}_F)^2 + (\mathcal{M}-2)^2\hat{C}^{(1)}_F.
\end{aligned} \tag{5·27}$$

In this way every Casimir operators $\hat{C}^{(k)}_E$ are reduced to $\hat{C}^{(1)}_F$ and $\hat{C}^{(2)}_F$. Hence we have ascertained that the irreducible representation of the E-$u(\mathcal{M})$ algebra is completely prescribed in terms of that of the retracted F-$u(2)$ algebra.

Now we derive explicitly the irreducible representation of the E-$u(\mathcal{M})$ algebra with the use of the irreducible representation of the retracted F-$u(2)$ algebra. First we note that the retracted F-$u(2)$ algebra is expressed as a direct sum of $u(2)$ algebra as follows:

$$\hat{T}_+ = \sum_{p=1}^{\mathcal{M}} \hat{T}_{p+}, \quad \hat{T}_- = \sum_{p=1}^{\mathcal{M}} \hat{T}_{p-}, \quad \hat{T}_0 = \sum_{p=1}^{\mathcal{M}} \hat{T}_{p0}, \quad \hat{N} = \sum_{p=1}^{\mathcal{M}} \hat{N}_p, \tag{5·28a}$$

$$\hat{T}_{p+} = \hat{x}^\dagger_{1p}\hat{x}_{2p}, \quad \hat{T}_{p-} = \hat{x}^\dagger_{2p}\hat{x}_{1p}, \quad \hat{T}_{p0} = \frac{1}{2}(\hat{x}^\dagger_{1p}\hat{x}_{1p} - \hat{x}^\dagger_{2p}\hat{x}_{2p}), \tag{5·28b}$$

$$\hat{N}_p = \sum_{a=1}^{2} \hat{x}^\dagger_{ap}\hat{x}_{ap}. \tag{5·28c}$$

The Casimir operators for the retracted F-$u(2)$ and the retracted ${}^p u(2)$ algebra are expressed in the form

$$\begin{aligned}
\hat{C}^{(1)}_F &= \hat{N}, \\
\hat{C}^{(2)}_F &= 2\hat{\boldsymbol{T}}^2 + \frac{1}{2}\hat{N}^2, \quad \hat{\boldsymbol{T}}^2 = \frac{1}{2}(\hat{T}_+\hat{T}_- + \hat{T}_-\hat{T}_+) + \hat{T}_0{}^2,
\end{aligned} \tag{5·29a}$$

$${}^p\hat{C}^{(1)} = \hat{N}_p. \tag{5·29b}$$

We note the following relation:

$$\begin{aligned}\hat{\boldsymbol{T}}_p{}^2 &= \frac{1}{2}(\hat{T}_{p+}\hat{T}_{p-} + \hat{T}_{p-}\hat{T}_{p+}) + \hat{T}_{p0}{}^2 \\ &= \frac{1}{2}\hat{N}_p\left(\frac{1}{2}\hat{N}_p + 1\right).\end{aligned} \tag{5·30}$$

The irreducible representation can be specified in terms of the eigenvalues of Casimir operators $\{\hat{C}_F^{(1)}, \hat{C}_F^{(2)}, ({}^p\hat{C}^{(1)}; p = 1, 2, \cdots, \mathcal{M})\}$. Further, it is easily ascertained that the following operators are mutually commutable:

$$\{\hat{\boldsymbol{T}}^2, \hat{T}_0, (\hat{N}_p; p = 1, 2, \cdots, \mathcal{M}), \hat{N}\}. \tag{5·31}$$

Now we can give the orthonormal basis of the representation carrier space as follows: With the use of the clusters

$$[\hat{x}_p^\dagger]^{[l_p, m_p]} = \frac{1}{\sqrt{(l_p + m_p)!(l_p - m_p)!}}(\hat{x}_{1p}^\dagger)^{l_p + m_p}(\hat{x}_{2p}^\dagger)^{l_p - m_p}, \quad (p = 1, 2, \cdots, \mathcal{M}) \tag{5·32}$$

we define cluster states

$$|[l_p, m_p]\rangle = [\hat{x}_p^\dagger]^{[l_p, m_p]}|0\rangle_x. \quad (\hat{x}_{ap}|0\rangle_x = 0; a = 1, 2, p = 1, 2, \cdots, \mathcal{M}) \tag{5·33}$$

These cluster states satisfy the relation

$$\begin{aligned}\hat{T}_{p+}|[l_p, m_p]\rangle &= \sqrt{(l_p - m_p)(l_p + m_p + 1)}|[l_p, m_p + 1]\rangle, \\ \hat{T}_{p-}|[l_p, m_p]\rangle &= \sqrt{(l_p + m_p)(l_p - m_p + 1)}|[l_p, m_p - 1]\rangle, \\ \hat{T}_{p0}|[l_p, m_p]\rangle &= m_p|[l_p, m_p]\rangle, \\ \hat{\boldsymbol{T}}_p{}^2|[l_p, m_p]\rangle &= l_p(l_p + 1)|[l_p, m_p]\rangle, \\ \hat{N}_p|[l_p, m_p]\rangle &= 2l_p|[l_p, m_p]\rangle.\end{aligned} \tag{5·34}$$

The above cluster states are nothing but the Schwinger representation of the $su(2)$ algebra,[7] that is, the spinor representation of the $su(2)$ algebra. The basis of the irreducible representation carrier space is given by the products of the clusters:

$$\left|\prod_{p=1}^{\mathcal{M}}[l_p, m_p]\right\rangle = \prod_{p=1}^{\mathcal{M}}[\hat{x}_p^\dagger]^{[l_p, m_p]}|0\rangle_x. \tag{5·35}$$

The actions of $\hat{E}_q^p$ on the above state are given by

$$\hat{E}_p^p\left|\prod_{p=1}^{\mathcal{M}}[l_p, m_p]\right\rangle = 2l_p\left|\prod_{p=1}^{\mathcal{M}}[l_p, m_p]\right\rangle, \tag{5·36a}$$

$$\hat{E}^p_q \Big| \prod_{p=1}^{\mathcal{M}} [l_p, m_p] \Big\rangle$$
$$= \sqrt{(l_p + m_p + 1)(l_q + m_q)} \Big| \Big[l_p + \frac{1}{2}, m_q + \frac{1}{2} \Big] \Big[l_q - \frac{1}{2}, m_q - \frac{1}{2} \Big] \prod_{r(\neq p,q)} [l_r, m_r] \Big\rangle$$
$$+ \sqrt{(l_p - m_p + 1)(l_q - m_q)} \Big| \Big[l_p + \frac{1}{2}, m_p - \frac{1}{2} \Big] \Big[l_q - \frac{1}{2}, m_q + \frac{1}{2} \Big] \prod_{r(\neq p,q)} [l_r, m_r] \Big\rangle;$$
$$p \neq q. \tag{5·36b}$$

The space spanned by the above basis vectors is a $u(2)$ product space. Therefore, it is apparent how to construct the basis of the irreducible representation carrier space. For instance, the basis is provided with the aid of the Clebsch-Gordan coefficient as

$$|l_1 l_2 \cdots l_{\mathcal{M}} (L_{12} L_{123} \cdots L_{12\cdots(\mathcal{M}-1)}) : L L_0\rangle$$
$$= \sum_{m_1 m_2 \cdots m_{\mathcal{M}}} (l_1 m_1 l_2 m_2 | L_{12} M_{12}) (L_{12} M_{12} l_3 m_3 | L_{123} M_{123}) \cdots$$
$$\times (L_{12\cdots(\mathcal{M}-1)} M_{12\cdots(\mathcal{M}-1)} l_{\mathcal{M}} m_{\mathcal{M}} | L L_0) \Big| \prod_{p=1}^{\mathcal{M}} [l_p, m_p] \Big\rangle, \tag{5·37a}$$
$$\Big(M_{12} = m_1 + m_2, M_{123} = M_{12} + m_3, \cdots, L_0 = \sum_{p=1}^{\mathcal{M}} m_p \Big), \tag{5·37b}$$

which satisfies the relations

$$\hat{\boldsymbol{T}}^2 |l_1 l_2 \cdots l_{\mathcal{M}} (L_{12} L_{123} \cdots L_{12\cdots(\mathcal{M}-1)}) : L L_0\rangle$$
$$= L(L+1) |l_1 l_2 \cdots l_{\mathcal{M}} (L_{12} L_{123} \cdots L_{12\cdots(\mathcal{M}-1)}) : L L_0\rangle,$$
$$\hat{T}_0 |l_1 l_2 \cdots l_{\mathcal{M}} (L_{12} L_{123} \cdots L_{12\cdots(\mathcal{M}-1)}) : L L_0\rangle$$
$$= L_0 |l_1 l_2 \cdots l_{\mathcal{M}} (L_{12} L_{123} \cdots L_{12\cdots(\mathcal{M}-1)}) : L L_0\rangle,$$
$$\hat{N}_p |l_1 l_2 \cdots l_{\mathcal{M}} (L_{12} L_{123} \cdots L_{12\cdots(\mathcal{M}-1)}) : L L_0\rangle$$
$$= 2 l_p |l_1 l_2 \cdots l_{\mathcal{M}} (L_{12} L_{123} \cdots L_{12\cdots(\mathcal{M}-1)}) : L L_0\rangle,$$
$$\hat{N} |l_1 l_2 \cdots l_{\mathcal{M}} (L_{12} L_{123} \cdots L_{12\cdots(\mathcal{M}-1)}) : L L_0\rangle$$
$$= \Big(\sum_{p=1}^{\mathcal{M}} 2 l_p \Big) |l_1 l_2 \cdots l_{\mathcal{M}} (L_{12} L_{123} \cdots L_{12\cdots(\mathcal{M}-1)}) : L L_0\rangle. \tag{5·38}$$

The action $\hat{F}^a_a$ on the above state is given by

$$\hat{F}^1_1 |l_1 l_2 \cdots l_{\mathcal{M}} (L_{12} L_{123} \cdots L_{12\cdots(\mathcal{M}-1)}) : L L_0\rangle$$
$$= \Big(\sum_{p=1}^{\mathcal{M}} l_p - L_0 \Big) |l_1 l_2 \cdots l_{\mathcal{M}} (L_{12} L_{123} \cdots L_{12\cdots(\mathcal{M}-1)}) : L L_0\rangle$$
$$= n_1 |l_1 l_2 \cdots l_{\mathcal{M}} (L_{12} L_{123} \cdots L_{12\cdots(\mathcal{M}-1)}) : L L_0\rangle,$$

$$\begin{aligned}
&\hat{F}_2^2|l_1l_2\cdots l_{\mathcal{M}}(L_{12}L_{123}\cdots L_{12\cdots(\mathcal{M}-1)}):LL_0\rangle \\
&\quad=\Big(\sum_{p=1}^{\mathcal{M}} l_p+L_0\Big)|l_1l_2\cdots l_{\mathcal{M}}(L_{12}L_{123}\cdots L_{12\cdots(\mathcal{M}-1)}):LL_0\rangle \\
&\quad=n_2|l_1l_2\cdots l_{\mathcal{M}}(L_{12}L_{123}\cdots L_{12\cdots(\mathcal{M}-1)}):LL_0\rangle. \qquad (5\cdot39)
\end{aligned}$$

The action of $\hat{E}_p^p$ on the irreducible representation carrier space is

$$\begin{aligned}
&\hat{E}_p^p|l_1l_2\cdots l_{\mathcal{M}}(L_{12}L_{123}\cdots L_{12\cdots(\mathcal{M}-1)}):LL_0\rangle \\
&\qquad=2l_p|l_1l_2\cdots l_{\mathcal{M}}(L_{12}L_{123}\cdots L_{12\cdots(\mathcal{M}-1)}):LL_0\rangle. \qquad (5\cdot40)
\end{aligned}$$

On the other hand, we need various types of recoupling rule of angular momentum and of many-j symbol such as the Racah coefficient for the description of the action of $\hat{E}_q^p(p\neq q)$.

§6. $u(2)$ algebra

In order to illustrate the retraction of the algebra, we take up a $u(2)$ algebra associated with the Lipkin model, i.e., a case $\mathcal{M}=2$.

6.1. *General treatment of $u(2)$ algebra*

The E- and the F-$u(2)$ algebra are described respectively by

$$\hat{E}_q^p=\sum_{a=1}^{2}\hat{x}_{ap}^{\dagger}\hat{x}_{aq},\quad \hat{F}_b^a=\sum_{p=1}^{2}\hat{x}_{ap}^{\dagger}\hat{x}_{bp}. \qquad (6\cdot1)$$

According to the conventional treatment, we use the following notations:

$$\begin{array}{lll}
\hat{S}_+=\hat{E}_1^2, & \hat{S}_-=\hat{E}_2^1, & \hat{S}_0=\frac{1}{2}(\hat{E}_2^2-\hat{E}_1^1), \\
\hat{T}_+=\hat{F}_1^2, & \hat{T}_-=\hat{F}_2^1, & \hat{T}_0=\frac{1}{2}(\hat{F}_2^2-\hat{F}_1^1), \\
\hat{N}_p=\hat{E}_p^p, & \hat{n}_a=\hat{F}_a^a, & \hat{N}=\sum_{p=1}^{2}\hat{N}_p=\sum_{a=1}^{2}\hat{n}_a,
\end{array} \qquad (6\cdot2)$$

$$\begin{aligned}
\hat{\boldsymbol{S}}^2&=\frac{1}{2}(\hat{S}_+\hat{S}_-+\hat{S}_-\hat{S}_+)+\hat{S}_0{}^2=\frac{1}{2}(\hat{T}_+\hat{T}_-+\hat{T}_-\hat{T}_+)+\hat{T}_0{}^2 \\
&=\hat{\boldsymbol{T}}^2. \qquad (6\cdot3)
\end{aligned}$$

In the previous section, we have shown that the irreducible representation carrier space of the E-$u(2)$ algebra is prescribed by the set of eigenvalues of the operators

$$\{\hat{\boldsymbol{S}}^2,\hat{S}_0,(\hat{F}_a^a;a=1,2)\}. \qquad (6\cdot4)$$

This is shown by another way as follows: The E-$u(2)$ algebra is a direct sum of $u(2)$ algebras, $\bigoplus_{a=1}^{2}{}_a u(2)$, i.e.,

$$\hat{S}_j=\sum_{a=1}^{2}\hat{S}_{aj};\quad j=\pm,0, \qquad (6\cdot5\text{a})$$

$$\hat{S}_{a+} = \hat{x}^\dagger_{a2}\hat{x}_{a1}, \quad \hat{S}_{a-} = \hat{x}^\dagger_{a1}\hat{x}_{a2}, \quad \hat{S}_{a0} = \frac{1}{2}(\hat{x}^\dagger_{a2}\hat{x}_{a2} - \hat{x}^\dagger_{a1}\hat{a}_{a1}), \tag{6·5b}$$

the Casimir operator of which is

$$\begin{aligned}\hat{\boldsymbol{S}}_a{}^2 &= \frac{1}{2}(\hat{S}_{a+}\hat{S}_{a-} + \hat{S}_{a-}\hat{S}_{a+}) + \hat{S}_{a0}{}^2 \\ &= \frac{1}{2}\hat{F}^a_a\left(\frac{1}{2}\hat{F}^a_a + 1\right).\end{aligned} \tag{6·6}$$

Then we have the relations

$$[\hat{\boldsymbol{S}}_a{}^2, \hat{S}_{bj}]_- = [\hat{\boldsymbol{S}}_a{}^2, \hat{S}_j]_- = [\hat{\boldsymbol{S}}_a{}^2, \hat{\boldsymbol{S}}^2]_- = 0. \quad (j = \pm, 0) \tag{6·7}$$

These relations imply that the set of operators (6·4) is necessary and sufficient to the specification of the irreducible representation carrier space.

Hence, with the aid of the clusters

$$[\hat{x}^\dagger_a]^{[\frac{n_a}{2}m_a]} = \frac{1}{\sqrt{(\frac{n_a}{2} - m_a)!(\frac{n_a}{2} + m_a)!}}(\hat{x}^\dagger_{a1})^{\frac{n_a}{2}-m_a}(\hat{x}^\dagger_{a2})^{\frac{n_a}{2}+m_a}, \quad (a = 1, 2) \tag{6·8}$$

the orthonormal basis vector of irreducible representation carrier space of the E-$u(2)$ algebra is given as follows:

$$\left|\frac{n_1}{2}\frac{n_2}{2} : SS_0\right\rangle = \sum_{m_1=-n_1/2}^{n_1/2}\sum_{n_2=-n_2/2}^{n_2/2}\left(\frac{n_1}{2}m_1\frac{n_2}{2}m_2\middle|SS_0\right)\left|\left[\frac{n_1}{2}m_1\right]\left[\frac{n_2}{2}m_2\right]\right\rangle, \tag{6·9a}$$

$$\left\langle\frac{n_1}{2}\frac{n_2}{2} : SS_0\middle|\frac{n'_1}{2}\frac{n'_2}{2} : S'S'_0\right\rangle = \delta_{n_1n'_1}\delta_{n_2n'_2}\delta_{SS'}\delta_{S_0S'_0}, \tag{6·9b}$$

where we have used the notation

$$\left|\left[\frac{n_1}{2}m_1\right]\left[\frac{n_2}{2}m_2\right]\right\rangle = [\hat{x}^\dagger_1]^{[\frac{n_1}{2}m_1]}[\hat{x}^\dagger_2]^{[\frac{n_2}{2}m_2]}|0\rangle_x. \tag{6·9c}$$

Then the state vector satisfies the relations

$$\begin{aligned}
\hat{S}_\pm\left|\frac{n_1}{2}\frac{n_2}{2} : SS_0\right\rangle &= \sqrt{(S \pm S_0 + 1)(S \mp S_0)}\left|\frac{n_1}{2}\frac{n_2}{2} : SS_0 \pm 1\right\rangle, \\
\hat{S}_0\left|\frac{n_1}{2}\frac{n_2}{2} : SS_0\right\rangle &= S_0\left|\frac{n_1}{2}\frac{n_2}{2} : SS_0\right\rangle, \\
\hat{\boldsymbol{S}}^2\left|\frac{n_1}{2}\frac{n_2}{2} : SS_0\right\rangle &= S(S+1)\left|\frac{n_1}{2}\frac{n_2}{2} : SS_0\right\rangle, \\
\hat{n}_a\left|\frac{n_1}{2}\frac{n_2}{2} : SS_0\right\rangle &= n_a\left|\frac{n_1}{2}\frac{n_2}{2} : SS_0\right\rangle, \quad (a = 1, 2) \\
\hat{N}\left|\frac{n_1}{2}\frac{n_2}{2} : SS_0\right\rangle &= (n_1 + n_2)\left|\frac{n_1}{2}\frac{n_2}{2} : SS_0\right\rangle.
\end{aligned} \tag{6·10}$$

With the use of the canonical transformation

$$\hat{x}_{12}^\dagger = \hat{X}_{12}^\dagger \exp[i\hat{\phi}_1], \quad \hat{x}_{11}^\dagger = \sqrt{\hat{n}_1 - \hat{X}_{12}^\dagger \hat{X}_{12}} \exp[i\hat{\phi}_1],$$
$$\hat{x}_{21}^\dagger = \hat{X}_{21}^\dagger \exp[i\hat{\phi}_2], \quad \hat{x}_{22}^\dagger = \sqrt{\hat{n}_2 - \hat{X}_{21}^\dagger \hat{X}_{21}} \exp[i\hat{\phi}_2],$$
$$\text{and their hermite conjugate,} \tag{6·11}$$

the state vector is rewritten in the form

$$\left|\frac{n_1}{2}\frac{n_2}{2} : SS_0\right\rangle = \exp[i(n_1\hat{\phi}_1 + n_2\hat{\phi}_2)]\left|\left(\frac{n_1}{2}\frac{n_2}{2}\right) : SS_0\right\rangle, \tag{6·12a}$$

$$\left|\left(\frac{n_1}{2}\frac{n_2}{2}\right) : SS_0\right\rangle = \sum_{m_1,m_2}\left(\frac{n_1}{2}m_1\frac{n_2}{2}m_2\middle|SS_0\right)\frac{1}{\sqrt{(\frac{1}{2}n_1+m_1)!(\frac{1}{2}n_2-m_2)!}} \times (\hat{X}_{12}^\dagger)^{\frac{1}{2}n_1+m_1}(\hat{X}_{21}^\dagger)^{\frac{1}{2}n_2-m_2}|0\rangle_X, \tag{6·12b}$$

$$\hat{X}_{12}|0\rangle_X = \hat{X}_{21}|0\rangle_X = 0, \tag{6·12c}$$

and the generators can be expressed as follows:

$$\hat{S}_+ = \hat{X}_{12}^\dagger\sqrt{\hat{n}_1 - \hat{X}_{12}^\dagger\hat{X}_{12}} + \sqrt{\hat{n}_2 - \hat{X}_{21}^\dagger\hat{X}_{21}}\ \hat{X}_{21},$$
$$\hat{S}_- = \sqrt{\hat{n}_1 - \hat{X}_{12}^\dagger\hat{X}_{12}}\ \hat{X}_{12} + \hat{X}_{21}^\dagger\sqrt{\hat{n}_2 - \hat{X}_{21}^\dagger\hat{X}_{21}},$$
$$\hat{S}_0 = \left(\hat{X}_{12}^\dagger\hat{X}_{12} - \frac{1}{2}\hat{n}_1\right) - \left(\hat{X}_{21}^\dagger\hat{X}_{21} - \frac{1}{2}\hat{n}_2\right). \tag{6·13}$$

The action of these generators on the above vector does not affect the phase factor.

On the other hand, the irreducible representation of the present model is specified only two quantities (S, N),[3] where they are given in terms of the seniority number ν_a of each single-particle level as

$$S = \Omega - \frac{1}{2}(\nu_1 + \nu_2), \quad N = 2\Omega - \nu_1 + \nu_2, \tag{6·14}$$

and also we obtain the expression for the eigenvalue of $\hat{n}_a$

$$n_1 = 2\Omega - \nu_1 - n_0, \quad n_2 = \nu_2 + n_0. \tag{6·15}$$

For fixed (n_1, n_2), the expression (6·9c) is an extension of the spinor representation.

Here we give the irreducible representation based on the F-$u(2)$ algebra. In this case, the Casimir operators are given by

$$\{\hat{C}_F^{(1)}, \hat{C}_F^{(2)}, ({}^p\hat{C}^{(1)}; p = 1, 2)\}, \tag{6·16}$$

where the first two operators are equivalent to those of the E-$u(2)$ algebra and the third ones are expressed as

$$ {}^{p}\hat{C}^{(1)} = \hat{N}_p. \quad (p = 1, 2) \tag{6·17} $$

Then with the use of clusters

$$ [\hat{x}_p^\dagger]^{[\frac{N_p}{2} m_p]} = \frac{1}{\sqrt{(\frac{1}{2}N_p - m_p)!(\frac{1}{2}N_p + m_p)!}} (\hat{x}_{1p}^\dagger)^{\frac{1}{2}N_p - m_p} (\hat{x}_{2p}^\dagger)^{\frac{1}{2}N_p + m_p}, \quad (p = 1, 2) \tag{6·18} $$

the orthonormal basis vectors of irreducible carrier space of the E-$u(2)$ algebra are given as follows:

$$ \left| \frac{N_1}{2} \frac{N_2}{2} : SM \right\rangle = \sum_{m_1=-N_1/2}^{N_1/2} \sum_{m_2=-N_2/2}^{N_2/2} \left(\frac{N_1}{2} m_1 \frac{N_2}{2} m_2 \middle| SM \right) \left| \left[\frac{N_1}{2} m_1 \right] \left[\frac{N_2}{2} m_2 \right] \right\rangle, \tag{6·19a} $$

$$ \left\langle \frac{N_1}{2} \frac{N_2}{2} : SM \middle| \frac{N_1'}{2} \frac{N_2'}{2} : S'M' \right\rangle = \delta_{N_1 N_1'} \delta_{N_2 N_2'} \delta_{SS'} \delta_{MM'}, \tag{6·19b} $$

where we have used the notation

$$ \left| \left[\frac{N_1}{2} m_1 \right] \left[\frac{N_2}{2} m_2 \right] \right\rangle = [\hat{x}_1^\dagger]^{[\frac{N_1}{2} m_1]} [\hat{x}_2^\dagger]^{[\frac{N_2}{2} m_2]} |0\rangle_x. \tag{6·19c} $$

Here we note the relations

$$ \hat{N}_p \left| \frac{N_1}{2} \frac{N_2}{2} : SM \right\rangle = N_p \left| \frac{N_1}{2} \frac{N_2}{2} : SM \right\rangle, $$

$$ \hat{S}_0 = \frac{1}{2}(\hat{N}_2 - \hat{N}_1). \tag{6·20} $$

The above basis vectors should satisfy the relations

$$ \begin{aligned} \hat{S}_0 \left| \frac{N_1}{2} \frac{N_2}{2} : SM \right\rangle &= S_0 \left| \frac{N_1}{2} \frac{N_2}{2} : SM \right\rangle, \quad \left(S_0 = \frac{1}{2}(N_2 - N_1) \right) \\ \hat{S}_\pm \left| \frac{N_1}{2} \frac{N_2}{2} : SM \right\rangle &= \sqrt{S \pm S_0 + 1)(S \mp S_0)} \left| \frac{N_1 \mp 1}{2} \frac{N_2 \pm 1}{2} : SM \right\rangle, \\ \hat{\boldsymbol{S}}^2 \left| \frac{N_1}{2} \frac{N_2}{2} : SM \right\rangle &= S(S+1) \left| \frac{N_1}{2} \frac{N_2}{2} : SM \right\rangle. \end{aligned} \tag{6·21} $$

As the quantity N_p expresses the nucleon number in the single-particle level p, the physical meaning of the actions $\hat{S}_\pm$ is very clear.

Further, we can get some formula about the Clebsch-Gordan coefficient from above relations. For instance, the relation

$$ \begin{aligned} &\sqrt{(j_1 + m_1 + 1)(j_2 + m_2)} \left(j_1 + \frac{1}{2} m_1 + \frac{1}{2} j_2 - \frac{1}{2} m_2 - \frac{1}{2} \middle| JM \right) \\ &\quad + \sqrt{(j_1 - m_1 + 1)(j_2 - m_2)} \left(j_1 + \frac{1}{2} m_1 - \frac{1}{2} j_2 - \frac{1}{2} m_2 + \frac{1}{2} \middle| JM \right) \\ &= \sqrt{(J - j_1 + j_2)(J + j_1 - j_2 + 1)} (j_1 m_1 j_2 m_2 | JM), \end{aligned} \tag{6·22} $$

can be easily obtained as follows: First we note the relation

$$
\begin{aligned}
&\hat{S}_-\left|\left[\frac{N_1}{2}m_1\right]\left[\frac{N_2}{2}m_2\right]\right\rangle \\
&\quad= (\hat{x}_{11}^\dagger\hat{x}_{12}+\hat{x}_{21}^\dagger\hat{x}_{22})[x_1^\dagger]^{[\frac{N_1}{2}m_1]}[x_2^\dagger]^{[\frac{N_2}{2}m_2]}|0\rangle_x \\
&\quad= \sqrt{\left(\frac{1}{2}N_1+m_1+1\right)\left(\frac{1}{2}N_2+m_2\right)}\left|\left[\frac{N_1+1}{2}m_1+\frac{1}{2}\right]\left[\frac{N_2-1}{2}m_2-\frac{1}{2}\right]\right\rangle \\
&\qquad+\sqrt{\left(\frac{1}{2}N_1-m_1+1\right)\left(\frac{1}{2}N_2-m_2\right)}\left|\left[\frac{N_1+1}{2}m_1-\frac{1}{2}\right]\left[\frac{N_2-1}{2}m_2+\frac{1}{2}\right]\right\rangle \\
&\quad= \sum_S\left\{\sqrt{\left(\frac{1}{2}N_1+m_1+1\right)\left(\frac{1}{2}N_2+m_2\right)}\left(\frac{N_1+1}{2}\,m_1+\frac{1}{2}\,\frac{N_2-1}{2}\,m_2-\frac{1}{2}\middle|SM\right)\right. \\
&\qquad\left.+\sqrt{\left(\frac{1}{2}N_1-m_1+1\right)\left(\frac{1}{2}N_2-m_2\right)}\left(\frac{N_1+1}{2}\,m_1-\frac{1}{2}\,\frac{N_2-1}{2}\,m_2+\frac{1}{2}\middle|SM\right)\right\} \\
&\qquad\qquad\times\left|\frac{N_1+1}{2}\frac{N_2-1}{2}:SM\right\rangle;\qquad M=\frac{1}{2}(m_1-m_2).
\end{aligned}
$$

On the other hand, from the second relation of Eq. (6·21), we have the relation

$$
\begin{aligned}
&\hat{S}_-\left|\left[\frac{N_1}{2}m_1\right]\left[\frac{N_2}{2}m_2\right]\right\rangle \\
&\quad= \sum_S\left(\frac{N_1}{2}\,m_1\,\frac{N_2}{2}\,m_2\middle|SM\right)\hat{S}_-\left|\frac{N_1}{2}\frac{N_1}{2}:SM\right\rangle \\
&\quad= \sum_S\sqrt{(S-S_0+1)(S+S_0)}\left(\frac{N_1}{2}\,m_1\,\frac{N_2}{2}\,m_2\middle|SM\right)\left|\frac{N_1+1}{2}\frac{N_2-1}{2}:SM\right\rangle; \\
&\qquad\qquad S_0=\frac{1}{2}(N_2-N_1).
\end{aligned}
$$

Comparing both the relations obtained above, we have the relation

$$
\begin{aligned}
&\sqrt{\left(\frac{1}{2}N_1+m_1+1\right)\left(\frac{1}{2}N_2+m_2\right)}\left(\frac{N_1+1}{2}\,m_1+\frac{1}{2}\,\frac{N_2-1}{2}\,m_2-\frac{1}{2}\middle|SM\right) \\
&\quad+\sqrt{\left(\frac{1}{2}N_1-m_1+1\right)\left(\frac{1}{2}N_2-m_2\right)}\left(\frac{N_1+1}{2}\,m_1-\frac{1}{2}\,\frac{N_2-1}{2}\,m_2+\frac{1}{2}\middle|SM\right) \\
&\qquad=\sqrt{(S-S_0+1)(S+S_0)}\left(\frac{N_1}{2}\,m_1\,\frac{N_2}{2}\,m_2\middle|SM\right).
\end{aligned}
$$

With the replacement

$$
\left(\frac{N_1}{2},\frac{N_2}{2},S,S_0\right)\longrightarrow(j_1,j_2,J,j_2-j_1),
$$

we have the relation (6·22).

At the end of this subsection, we discuss the constraints (2·25). We have modified the constraints into the F-$u(2)$ algebra commutable with the E-$u(2)$ algebra. Hence

there involves a certain redundant degree of freedom. In the above treatment, the state $|\frac{N_1}{2}\frac{N_2}{2} : SM\rangle$ involves the meaningless quantum number M. Here we explicitly take into account the constraints, which are given in the present case as

$$\begin{aligned}\chi &= \left(X_{12}^*\sqrt{n_2 - X_{21}^*X_{21}} + X_{21}\sqrt{n_1 - X_{12}^*X_{12}}\right)\exp[i(\phi_1 - \phi_2)] = 0,\\ \chi^* &= \left(X_{12}\sqrt{n_2 - X_{21}^*X_{21}} + X_{21}^*\sqrt{n_1 - X_{12}^*X_{12}}\right)\exp[-i(\phi_1 - \phi_2)] = 0.\end{aligned} \tag{6·23}$$

In the classical version, the quasi-spin operators (6·13) can be given by

$$\begin{aligned}S_+ &= X_{12}^*\sqrt{n_1 - X_{12}^*X_{12}} + \sqrt{n_2 - X_{21}^*X_{21}}\ X_{21},\\ S_- &= \sqrt{n_1 - X_{12}^*X_{12}}\ X_{12} + X_{21}^*\sqrt{n_2 - X_{21}^*X_{21}},\\ S_0 &= \left(X_{12}^*X_{12} - \frac{1}{2}n_1\right) - \left(X_{21}^*X_{21} - \frac{1}{2}n_2\right),\end{aligned} \tag{6·24}$$

and the amount of the quasi-spin can be expressed in the form

$$\begin{aligned}\boldsymbol{S}^2 &= S_+S_- + S_0{}^2\\ &= \frac{1}{4}(n_1 - n_2)^2 + \chi\chi^*.\end{aligned} \tag{6·25}$$

Therefore, under the constraints the amount of quasi-spin is uniquely specified as $\frac{1}{2}(n_1 - n_2)$ $(n_1 > n_2)$, contrary to the above treatment where $S = \frac{1}{2}(n_1 - n_2), \frac{1}{2}(n_1 - n_2) + 1, \cdots, \frac{1}{2}(n_1 + n_2)$.

By taking account of the constraints explicitly, the variables can be expressed as

$$X_{12}^* = \sqrt{\frac{n_1}{n_1 - n_2}}\ X^*, \quad X_{21}^* = -\sqrt{\frac{n_2}{n_1 - n_2}}\ X, \tag{6·26}$$

and the expression of the quasi-spin operators (6·29) is rewritten as

$$S_+ = X^*\sqrt{2S - X^*X}, \quad S_- = \sqrt{2S - X^*X}\ X, \quad S_0 = X^*X - S; \tag{6·27a}$$

$$S = \frac{1}{2}(n_1 - n_2). \tag{6·27b}$$

The quantized form is given by

$$\hat{S}_+ = \hat{X}^*\sqrt{2\hat{S} - \hat{X}^*\hat{X}}, \quad \hat{S}_- = \sqrt{2\hat{S} - \hat{X}^*\hat{X}}\ X, \quad \hat{S}_0 = \hat{X}^*\hat{X} - \hat{S}, \tag{6·28a}$$

$$\hat{\boldsymbol{S}}^2 = \hat{S}(\hat{S} + 1); \quad \hat{S} = \frac{1}{2}(\hat{n}_1 - \hat{n}_2). \tag{6·28b}$$

Therefore, under the constraints the quantities n_a are just related with the seniority:

$$n_1 = 2\Omega - \nu_1, \quad n_2 = \nu_2. \tag{6·29}$$

6.2. *Retracted $u(1)$ algebra*

Let us take up the case $(n_1, n_2) = (N, 0)$. In this case, we have the following retracted forms of generators:

$$\hat{E}_1^2 = \hat{x}_{12}^\dagger \hat{x}_{11}, \quad \hat{E}_2^1 = \hat{x}_{11}^\dagger \hat{x}_{12}, \quad \hat{E}_1^1 = \hat{x}_{11}^\dagger \hat{x}_{11}, \quad \hat{E}_2^2 = \hat{x}_{12}^\dagger \hat{x}_{12}, \tag{6·30a}$$

$$\hat{F}_1^1 = \hat{x}_{11}^\dagger \hat{x}_{11} + \hat{x}_{12}^\dagger \hat{x}_{12}. \tag{6·30b}$$

It is apparent that the F-$u(2)$ algebra is retracted to a $u(1)$ algebra, contrary to the E-$u(2)$ algebra. Following the usual treatment, we define the generators of $u(2)$

$$\hat{S}_+ = \hat{E}_1^2, \quad \hat{S}_- = \hat{E}_2^1, \quad \hat{S}_0 = \frac{1}{2}(\hat{E}_2^2 - \hat{E}_1^1), \quad \hat{N} = \hat{E}_1^1 + \hat{E}_2^2 = \hat{F}_1^1. \tag{6·31}$$

Due to the duality, the irreducible representation of the E-$u(2)$ algebra can be specified in terms of the eigenvalue N of the Casimir operator $\hat{C}_F^{(1)} (= \hat{N})$. The state vectors of irreducible representation carrier space are given by

$$\left|\frac{1}{2}N, m\right\rangle = \frac{1}{\sqrt{(\frac{1}{2}N - m)!(\frac{1}{2}N + m)!}} (\hat{x}_{11}^\dagger)^{\frac{1}{2}N - m} (\hat{x}_{12}^\dagger)^{\frac{1}{2}N + m} |0\rangle_x;$$

$$m = -\frac{1}{2}N, -\frac{1}{2}N + 1, \cdots, \frac{1}{2}N, \tag{6·32}$$

which satisfy the relations

$$\hat{N}\left|\frac{1}{2}N, m\right\rangle = N\left|\frac{1}{2}N, m\right\rangle, \quad \hat{S}_0\left|\frac{1}{2}N, m\right\rangle = m\left|\frac{1}{2}N, m\right\rangle. \tag{6·33}$$

On the other hand, the vectors of irreducible representation of the E-$u(2)$ algebra are specified in terms of eigenvalues (S, S_0). With the use of the relation

$$\begin{aligned} \hat{\boldsymbol{S}}^2 &= \frac{1}{2}(\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+) + \hat{S}_0{}^2 \\ &= \frac{1}{2}\hat{N}\left(\frac{1}{2}\hat{N} + 1\right), \end{aligned} \tag{6·34}$$

the eigenvalues (S, S_0) are just equal to $(\frac{1}{2}N, m)$, i.e.,

$$\hat{\boldsymbol{S}}^2\left|\frac{1}{2}N, m\right\rangle = \frac{1}{2}N\left(\frac{1}{2}N + 1\right)\left|\frac{1}{2}N, m\right\rangle, \quad \hat{S}_0\left|\frac{1}{2}N, m\right\rangle = m\left|\frac{1}{2}N, m\right\rangle. \tag{6·35}$$

In other words, the state (6·32) corresponds to a special case of $|\frac{N_1}{2}\frac{N_2}{2} : SM\rangle$ with $(S, M) = (\frac{1}{2}(N_1 + N_2), -\frac{1}{2}(N_1 + N_2))$, i.e.,

$$\left|\frac{1}{2}N, m\right\rangle = \left|\frac{1}{2}N_1 = \frac{1}{2}N - m, \frac{1}{2}N_2 = \frac{1}{2}N + m : S = \frac{1}{2}N, M = -S\right\rangle. \tag{6·36}$$

By the retraction of the algebra, the redundancy in the representation space can be removed.

When we introduce the canonical variables $\{(\hat{X}^*, \hat{X}), (\hat{N}, \hat{\phi})\}$ through the relations

$$\hat{x}_{11} = \exp[-i\hat{\phi}]\sqrt{\hat{N} - \hat{X}^\dagger \hat{X}}, \quad \hat{x}_{11}^\dagger = \sqrt{\hat{N} - \hat{X}^\dagger \hat{X}}\ \exp[i\hat{\phi}],$$
$$\hat{x}_{12} = \exp[-i\hat{\phi}]\hat{X}, \quad \hat{x}_{12}^\dagger = \hat{X}^\dagger \exp[i\hat{\phi}], \tag{6·37}$$

the state vector can be expressed as

$$\left|\frac{1}{2}N, m\right\rangle = \exp[iN\hat{\phi}]\frac{1}{\sqrt{(\frac{1}{2}N+m)!}}(\hat{X}^\dagger)^{\frac{1}{2}N+m}|0\rangle_X. \tag{6·38}$$

Further we have the Holstein-Primakoff representation:[6)]

$$\hat{S}_+ = \hat{X}^\dagger\sqrt{\hat{N} - \hat{X}^\dagger \hat{X}}, \quad \hat{S}_- = \sqrt{\hat{N} - \hat{X}^\dagger \hat{X}}\ \hat{X}, \quad \hat{S}_0 = \hat{X}^\dagger \hat{X} - \frac{1}{2}\hat{N}, \tag{6·39a}$$

$$\hat{\boldsymbol{S}}^2 = \frac{1}{2}\hat{N}\left(\frac{1}{2}\hat{N} + 1\right). \tag{6·39b}$$

By considering the original meaning of the quantity N and the relation (6·14), the present result can be applicable only to the case of the maximum of quasi-spin S for a given $N(\leq 2\Omega)$.

§7. $u(3)$ algebra

In this section, we discuss the irreducible representation of $u(3)$ algebra. The general treatment is the same as that of $u(\mathcal{M})$ algebra. The elements of the E- and the F-$u(3)$ algebra are given as follows:

$$\hat{E}_q^p = \sum_{a=1}^{3} \hat{x}_{ap}^\dagger \hat{x}_{aq}, \qquad \hat{F}_b^a = \sum_{p=1}^{3} \hat{x}_{ap}^\dagger \hat{x}_{bp}, \tag{7·1a}$$

$$[\hat{E}_q^p, \hat{E}_s^r]_- = \delta_{rq}\hat{E}_s^p - \delta_{ps}\hat{E}_q^r, \quad [\hat{F}_b^a, \hat{F}_d^c]_- = \delta_{cb}\hat{F}_d^a - \delta_{ad}\hat{F}_b^c, \tag{7·1b}$$

$$[\hat{E}_q^p, \hat{F}_b^a]_- = 0. \quad (a, b, c, d = 1, 2, 3, \quad p, q, r, s = 1, 2, 3) \tag{7·1c}$$

The Casimir operators of E-$u(3)$ and F-$u(3)$ algebra are respectively given by

$$\hat{C}_E^{(1)} = \sum_p \hat{E}_p^p = \hat{N}, \quad \hat{C}_E^{(2)} = \sum_{p_1,p_2} \hat{E}_{p_2}^{p_1}\hat{E}_{p_1}^{p_2}, \quad \hat{C}_E^{(3)} = \sum_{p_1,p_2,p_3} \hat{E}_{p_2}^{p_1}\hat{E}_{p_3}^{p_2}\hat{E}_{p_1}^{p_3},$$
$$\hat{C}_F^{(k)} = \hat{C}_E^{(k)}; \quad k = 1, 2, 3, \tag{7·2}$$

which satisfy the relations

$$[\hat{C}_E^{(k)}, \hat{E}_q^p]_- = [\hat{C}_E^{(k)}, \hat{F}_b^a]_- = 0. \tag{7·3}$$

Therefore the basis of the irreducible representation carrier space of the E-$u(3)$ algebra is specified by the set of eigenvalues of

$$\{(\hat{C}_E^{(k)}; k = 1, 2, 3),\ (\hat{F}_a^a; a = 1, 2, 3)\}. \tag{7·4}$$

With the use of clusters defined by

$$\Big[\sum_p n_{ap} = n_a\Big] = \prod_{p=1}^{3} \frac{1}{\sqrt{n_{ap}!}} (\hat{x}^\dagger_{ap})^{n_{ap}}, \quad (a = 1, 2, 3) \tag{7·5}$$

the basis is given by

$$\Big|\prod_{a=1}^{3} \Big[\sum_p n_{ap} = n_a\Big] : n\Big\rangle = \prod_{a=1}^{3} \Big[\sum_p n_{ap} = n_a\Big] |0\rangle_x. \quad \Big(\sum_{a=1}^{3} n_a = n\Big) \tag{7·6}$$

The operation of $\hat{E}^p_q$ is easily evaluated, for instance,

$$\hat{E}^1_1 \Big|\prod_{a=1}^{3} \Big[\sum_p n_{ap} = n_a\Big] : n\Big\rangle = \Big(\sum_{b=1}^{3} n_{b1}\Big) \Big|\prod_{a=1}^{3} \Big[\sum_p n_{ap} = n_a\Big] : n\Big\rangle, \tag{7·7a}$$

$$\begin{aligned} &\hat{E}^1_2 \Big|\prod_{a=1}^{3} \Big[\sum_p n_{ap} = n_a\Big] : n\Big\rangle \\ &\qquad = \sum_{b=1}^{3} \sqrt{(n_{b1}+1)n_{b2}} \Big|[(n_{b1}+1) + (n_{b2}-1) + n_{b3} = n_b] \\ &\qquad\qquad \times \prod_{a(\neq b)} \Big[\sum_p n_{ap} = n_a\Big] : n\Big\rangle. \end{aligned} \tag{7·7b}$$

On the other hand, the basis of the irreducible representation carrier space of the F-$u(3)$ algebra specified by the set of eigenvalues of

$$\{(\hat{C}^{(k)}_F; k = 1, 2, 3), (\hat{E}^p_p; p = 1, 2, 3)\} \tag{7·8}$$

also provides that of the E-$u(3)$ algebra. With the use of the clusters

$$\Big[\sum_a N_{ap} = N_p\Big] = \prod_{a=1}^{3} \frac{1}{\sqrt{N_{ap}!}} (\hat{x}^\dagger_{ap})^{N_{ap}}, \quad (p = 1, 2, 3) \tag{7·9}$$

the basis of the irreducible representation carrier space of F-$u(3)$ algebra is expressed in the following form:

$$\Big|\prod_{p=1}^{3} \Big[\sum_a N_{ap} = N_p\Big] : N\Big\rangle = \prod_{p=1}^{3} \Big[\sum_a N_{ap} = N_p\Big] |0\rangle_x, \quad \Big(\sum_p N_p = N\Big) \tag{7·10a}$$

$$x_{ap}|0\rangle_x = 0. \tag{7·10b}$$

The operation of $\hat{E}^p_q$ is easily obtained, for instance,

$$\hat{E}^1_1 \Big|\prod_{p=1}^{3} \Big[\sum_a N_{ap} = N_p\Big]\Big\rangle = N_1 \Big|\prod_{p=1}^{3} \Big[\sum_a N_{ap} = N_p\Big]\Big\rangle, \tag{7·11a}$$

$$\hat{E}_2^1 \Big| \prod_{p=1}^{3} \Big[\sum_a N_{ap} = N_p \Big] \Big\rangle$$
$$= \sum_{b=1}^{3} \sqrt{(N_{b1}+1)N_{b2}} \Big| \Big[\sum_{a(\neq b)} N_{a1} + (N_{b1}+1) = N_1 + 1 \Big]$$
$$\times \Big[\sum_{a(\neq b)} N_{a2} + (N_{b2}-1) = N_2 - 1 \Big] \Big[\sum_a N_{a3} = N_3 \Big] : N \Big\rangle. \quad (7\cdot 11\text{b})$$

As discussed in §6, this expression is favorable for making clear the physical ingredient of the action of $\hat{E}_q^p$, but it is tedious to evaluate the operation of $\hat{E}_q^p$. However, this form is very advantageous to the retractable F-$u(3)$ algebra.

7.1. *Retracted $u(2)$ algebra*

As has been shown in §4, the operators $\hat{F}_a^a$, whose eigenvalues are denoted by n_a, are commutable with $\hat{E}_q^p$ and then with the hamiltonian. Hence, we can rather freely settle the quantities n_a as the initial condition. In this and the next subsection, we investigate how to restrict the representation space by the initial condition. In the case $n_3 = 0$, the F-$u(3)$ algebra can be retracted to a $u(2)$ algebra:

$$\hat{E}_q^p = \sum_{a=1}^{2} \hat{x}_{ap}^\dagger \hat{x}_{aq}, \quad (7\cdot 12\text{a})$$

$$\hat{F}_b^a = \begin{cases} \sum_{p=1}^{3} \hat{x}_{ap}^\dagger \hat{x}_{bp}\,; & a, b = 1, 2, \\ 0 \qquad\qquad ; & \text{otherwise.} \end{cases} \quad (7\cdot 12\text{b})$$

The Casimir operators are given in the form:

$$\begin{aligned} \hat{C}_E^{(1)} &= \hat{N} = \hat{C}_F^{(1)}, \\ \hat{C}_E^{(2)} &= \hat{C}_F^{(2)} + \hat{C}_F^{(1)}, \\ \hat{C}_E^{(3)} &= \hat{C}_F^{(3)} + 2\hat{C}_F^{(2)} + \hat{C}_F^{(1)} \\ &= \frac{3}{2}\hat{C}_F^{(1)}\hat{C}_F^{(2)} + 3\hat{C}_F^{(2)} - \frac{1}{2}(\hat{C}_F^{(1)})^3 - \frac{1}{2}(\hat{C}_F^{(1)})^2 + \hat{C}_F^{(1)}. \end{aligned} \quad (7\cdot 13)$$

We define the following notations:

$$\hat{L}_+ = \hat{F}_2^1, \quad \hat{L}_- = \hat{F}_1^2, \quad \hat{L}_0 = \frac{1}{2}(\hat{F}_1^1 - \hat{F}_2^2), \quad (7\cdot 14\text{a})$$

$$\hat{N} = \hat{F}_1^1 + \hat{F}_2^2, \quad (7\cdot 14\text{b})$$

which satisfy the relations

$$[\hat{L}_+, \hat{L}_-] = 2\hat{L}_0, \quad [\hat{L}_0, \hat{L}_\pm] = \pm\hat{L}_\pm, \quad (7\cdot 15\text{a})$$

$$[\hat{N}, \hat{L}_j] = 0. \quad (j = 0, \pm) \quad (7\cdot 15\text{b})$$

Thus the retracted F-$u(2)$ algebra is decomposed into an $su(2)$ and a $u(1)$ algebra. The Casimir operators $\hat{C}_F^{(k)}$ can be expressed in the form

$$\hat{C}_F^{(1)} = \hat{N}, \quad \hat{C}_F^{(2)} = 2\hat{\boldsymbol{L}}^2 + \frac{1}{2}\hat{N}, \tag{7·16a}$$

$$\hat{\boldsymbol{L}}^2 = \frac{1}{2}(\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+) + \hat{L}_0{}^2. \tag{7·16b}$$

In this case, the irreducible representation can be easily obtained: First we notice the fact that the retracted $su(2)$ algebra can be expressed as a direct sum of three $su(2)$ algebras commutable with one another:

$$\hat{L}_j = \sum_{p=1}^{3} \hat{L}_{pj}, \quad (j = 0, \pm) \tag{7·17a}$$

$$\hat{L}_{p+} = \hat{x}_{1p}^\dagger \hat{x}_{2p}, \quad \hat{L}_{p-} = \hat{x}_{2p}^\dagger \hat{x}_{1p}, \quad \hat{L}_{p0} = \frac{1}{2}(\hat{x}_{1p}^\dagger \hat{x}_{1p} - \hat{x}_{2p}^\dagger \hat{x}_{2p}), \tag{7·17b}$$

$$[\hat{L}_{p+}, \hat{L}_{q-}]_- = 2\delta_{pq}\hat{L}_{p0}, \quad [\hat{L}_{p0}, \hat{L}_{q\pm}]_- = \pm\delta_{pq}\hat{L}_{p\pm}, \tag{7·18a}$$

$$\hat{\boldsymbol{L}}_p{}^2 = \frac{1}{2}(\hat{L}_{p+}\hat{L}_{p-} + \hat{L}_{p-}\hat{L}_{p+}) + \hat{L}_{p0}{}^2, \quad [\hat{\boldsymbol{L}}_p{}^2, \hat{L}_{qj}]_- = 0. \tag{7·18b}$$

Then we define the following basis vectors:

$$[l_p m_p] = \frac{1}{\sqrt{(l_p + m_p)!(l_p - m_p)!}}(\hat{x}_{1p}^\dagger)^{l_p+m_p}(\hat{x}_{2p}^\dagger)^{l_p-m_p}, \tag{7·19a}$$

$$\left|\prod_{p=1}^{3}[l_p m_p]\right\rangle = \prod_{p=1}^{3}[l_p m_p]|0\rangle_x, \tag{7·19b}$$

which satisfy the relations

$$\begin{aligned} \hat{\boldsymbol{L}}_q{}^2\left|\prod_{p=1}^{3}[l_p m_p]\right\rangle &= l_q(l_q+1)\left|\prod_{p=1}^{3}[l_p m_p]\right\rangle, \\ \hat{L}_{q0}\left|\prod_{p=1}^{3}[l_p m_p]\right\rangle &= m_q\left|\prod_{p=1}^{3}[l_p m_p]\right\rangle. \end{aligned} \tag{7·20}$$

Then the basis vector of the irreducible representation carrier space of F-$u(2)$ is given by

$$\begin{aligned} |l_1 l_2 (L_{12}) l_3 : LM\rangle = &\sum_{m_1, m_2, m_3, M_{12}} (l_1 m_1 l_2 m_2 | L_{12} M_{12})(L_{12} M_{12} l_3 m_3 | LM) \\ &\times \left|\prod_{p=1}^{3}[l_p m_p]\right\rangle, \end{aligned} \tag{7·21}$$

which satisfies the relations

$$\begin{aligned}
\hat{\boldsymbol{L}}_p^2|l_1l_2(L_{12})l_3:LM\rangle &= l_p(l_p+1)|l_1l_2(L_{12})l_3:LM\rangle,\\
\hat{\boldsymbol{L}}^2|l_1l_2(L_{12})l_3:LM\rangle &= L(L+1)|l_1l_2(L_{12})l_3:LM\rangle,\\
\hat{L}_0|l_1l_2(L_{12})l_3:LM\rangle &= M|l_1l_2(L_{12})l_3:LM\rangle,\\
\hat{N}|l_1l_2(L_{12})l_3:LM\rangle &= 2\sum_{p=1}^{3} l_p|l_1l_2(L_{12})l_3:LM\rangle.
\end{aligned} \tag{7·22}$$

We investigate the action of $\hat{E}_q^p$ on this state vector. First we take up the action of $\hat{E}_2^1$ on the state

$$|l_1l_2:LM\rangle \tag{7·23a}$$

defined by

$$|l_1l_2:LM\rangle = \sum_{m_1,m_2}(l_1m_1l_2m_2|LM)|[l_1m_1][l_2m_2]\rangle. \tag{7·23b}$$

$$\begin{aligned}
&\hat{E}_2^1|l_1l_2:LM\rangle\\
&\quad= \sum_{m_1,m_2}(l_1m_1l_2m_2|LM)\\
&\qquad\times\left\{\sqrt{(l_1+m_1+1)(l_2+m_2)}\left|\left[l_1+\frac{1}{2}\,m_1+\frac{1}{2}\right]\left[l_2-\frac{1}{2}\,m_2-\frac{1}{2}\right]\right\rangle\right.\\
&\qquad\left.+\sqrt{(l_1-m_1+1)(l_2-m_2)}\left|\left[l_1+\frac{1}{2}\,m_1-\frac{1}{2}\right]\left[l_2-\frac{1}{2}\,m_2+\frac{1}{2}\right]\right\rangle\right\}\\
&\quad= \sum_{m_1,m_2,L',M'}(l_1m_1l_2m_2|LM)|l_1l_2:L'M'\rangle\\
&\qquad\times\left\{\sqrt{(l_1+m_1+1)(l_2+m_2)}\left(l_1+\frac{1}{2}\,m_1+\frac{1}{2}\,l_2-\frac{1}{2}\,m_2-\frac{1}{2}\middle|L'M'\right)\right.\\
&\qquad\left.+\sqrt{(l_1-m_1+1)(l_2-m_2)}\left(l_1+\frac{1}{2}\,m_1-\frac{1}{2}\,l_2-\frac{1}{2}\,m_2+\frac{1}{2}\middle|L'M'\right)\right\}.
\end{aligned}$$

With the aid of the formula (6·22), we have the relation

$$\hat{E}_2^1|l_1l_2:LM\rangle = \sqrt{(L-l_1+l_2)(L+l_1-l_2+1)}\left|l_1+\frac{1}{2}\,l_2-\frac{1}{2}:LM\right\rangle.$$

In this way, we have the relations

$$\begin{aligned}
&\hat{E}_2^1|l_1l_2(L_{12})l_3:LM\rangle\\
&\qquad= \sqrt{(L_{12}-l_1+l_2)(L_{12}+l_1-l_2+1)}\left|l_1+\frac{1}{2}\,l_2-\frac{1}{2}(L_{12})l_3:LM\right\rangle,\\
&\hat{E}_1^2|l_1l_2(L_{12})l_3:LM\rangle\\
&\qquad= \sqrt{(L_{12}-l_2+l_1)(L_{12}+l_2-l_1+1)}\left|l_1-\frac{1}{2}\,l_2+\frac{1}{2}(L_{12})l_3:LM\right\rangle,\\
&\hat{E}_p^p|l_1l_2(L_{12})l_3:LM\rangle = 2l_p|l_1l_2(L_{12})l_3:LM\rangle. \quad (p=1,2,3)
\end{aligned} \tag{7·24}$$

Further the action of other $\hat{E}^p_q$ can be expressed in the following form:

$$
\begin{aligned}
&\hat{E}^3_1|l_1l_2(L_{12})l_3:LM\rangle \\
&=\sum_{L'_{12}}(-)^{l_1+l_2+l_3+L+2L_{12}}\sqrt{(2L_{12}+1)(L'_{12}+1)2l_1(2l_1+1)(2l_3+1)(2l_3+2)} \\
&\quad\times\left\{\begin{array}{ccc} l_1 & l_2 & L_{12} \\ L'_{12} & \frac{1}{2} & l_1-\frac{1}{2}\end{array}\right\}\left\{\begin{array}{ccc} L_{12} & l_3 & L \\ l_3+\frac{1}{2} & L'_{12} & \frac{1}{2}\end{array}\right\}\left|l_1-\frac{1}{2}\,l_2(L'_{12})l_3+\frac{1}{2}:LM\right\rangle, \\
&\hat{E}^1_3|l_1l_2l_3(L_{12}):LM\rangle \\
&=\sum_{L'_{12}}(-)^{l_1+l_2+l_3+L+2L'_{12}}\sqrt{(2L_{12}+1)(2L'_{12}+1)(2l_1+1)(2l_1+2)2l_3(2l_3+1)} \\
&\quad\times\left\{\begin{array}{ccc} l_1 & l_2 & L_{12} \\ L'_{12} & \frac{1}{2} & l_1+\frac{1}{2}\end{array}\right\}\left\{\begin{array}{ccc} L_{12} & l_3 & L \\ l_3-\frac{1}{2} & L'_{12} & \frac{1}{2}\end{array}\right\}\left|l_1+\frac{1}{2}\,l_2(L'_{12})l_3-\frac{1}{2}:LM\right\rangle, \\
&\hat{E}^3_2|l_1l_2(L_{12})l_3:LM\rangle \\
&=\sum_{L'_{12}}(-)^{l_1+l_2+l_3+L+L_{12}+L'_{12}+\frac{1}{2}}\sqrt{(2L_{12}+1)(2L'_{12}+1)2l_2(2l_2+1)(2l_3+1)(2l_3+2)} \\
&\quad\times\left\{\begin{array}{ccc} l_2 & l_1 & L_{12} \\ L'_{12} & \frac{1}{2} & l_2-\frac{1}{2}\end{array}\right\}\left\{\begin{array}{ccc} L_{12} & l_3 & L \\ l_3+\frac{1}{2} & L'_{12} & \frac{1}{2}\end{array}\right\}\left|l_1l_2-\frac{1}{2}(L'_{12})l_3+\frac{1}{2}:LM\right\rangle, \\
&\hat{E}^2_3|l_1l_2(L_{12})l_3:LM\rangle \\
&=\sum_{L'_{12}}(-)^{l_1+l_2+l_3+L+L_{12}+L'_{12}+\frac{1}{2}}\sqrt{(2L_{12}+1)(2L'_{12}+1)(2l_2+1)(2l_2+2)2l_3(2l_3+1)} \\
&\quad\times\left\{\begin{array}{ccc} l_2 & l_1 & L_{12} \\ L'_{12} & \frac{1}{2} & l_2+\frac{1}{2}\end{array}\right\}\left\{\begin{array}{ccc} L_{12} & l_3 & L \\ l_3-\frac{1}{2} & L'_{12} & \frac{1}{2}\end{array}\right\}\left|l_1l_2+\frac{1}{2}(L'_{12})l_3-\frac{1}{2}:LM\right\rangle,
\end{aligned}
\tag{7·25}
$$

where the notation $\{***\}$ denotes the 9-j symbol. The derivation of the above relations is given in the Appendix.

In the usual treatment of $su(3)$, the irreducible representation can be specified by the set of quantities $\{(\lambda,\mu),Y,I,I_0\}$, i.e., $|(\lambda,\mu),Y,I,I_0\rangle$, where Y, I and I_0 denote respectively the eigenvalues of the operators defined as follows:

$$\hat{I}_+=\hat{E}^1_2,\quad \hat{I}_-=\hat{E}^2_1,\quad \hat{I}_0=\frac{1}{2}(\hat{E}^1_1-\hat{E}^2_2), \tag{7·26a}$$

$$\hat{\boldsymbol{I}}^2=\frac{1}{2}(\hat{I}_+\hat{I}_-+\hat{I}_-\hat{I}_+)+\hat{I}_0{}^2, \tag{7·26b}$$

$$\hat{Y}=\frac{1}{3}(\hat{E}^1_1+\hat{E}^2_2-2\hat{E}^3_3), \tag{7·26c}$$

$$\hat{N}=\sum_{p=1}^{3}E^p_p. \tag{7·26d}$$

In the above expressions, $\hat{N}$ expresses the total number of nucleons in the present model and, in the quark model, it denotes the total number of quarks (u, d, s).

$$\begin{aligned}
\hat{\boldsymbol{I}}^2|(\lambda,\mu),Y,I,I_0\rangle &= I(I+1)|(\lambda,\mu),Y,I,I_0\rangle,\\
\hat{I}_0|(\lambda,\mu),Y,I,I_0\rangle &= I_0|(\lambda,\mu),Y,I,I_0\rangle,\\
\hat{Y}|(\lambda,\mu),Y,I,I_0\rangle &= Y|(\lambda,\mu),Y,I,I_0\rangle.
\end{aligned} \tag{7·27}$$

For the present representation, the action of these operators can be given as follows:

$$\begin{aligned}
&\hat{I}_+|l_1l_2(L_{12})l_3:LM\rangle\\
&\quad=\sqrt{(L_{12}-l_1+l_2)(L_{12}+l_1-l_2+1)}\left|l_1+\frac{1}{2}\,l_2-\frac{1}{2}(L_{12})l_3:LM\right\rangle,\\
&\hat{I}_-|l_1l_2(L_{12})l_3:LM\rangle\\
&\quad=\sqrt{(L_{12}+l_1-l_2)(L_{12}-l_1+l_2+1)}\left|l_1-\frac{1}{2}\,l_2+\frac{1}{2}(L_{12})l_3:LM\right\rangle,\\
&\hat{I}_0|l_1l_2(L_{12})l_3:LM\rangle\\
&\quad=(l_1-l_2)|l_1l_2(L_{12})l_3:LM\rangle,\\
&\hat{\boldsymbol{I}}^2|l_1l_2(L_{12})l_3:LM\rangle\\
&\quad=L_{12}(L_{12}+1)|l_1l_2(L_{12})l_3:LM\rangle,\\
&\hat{Y}|l_1l_2(L_{12})l_3:LM\rangle\\
&\quad=\frac{2}{3}(l_1+l_2-2l_3)|l_1l_2(L_{12})l_3:LM\rangle,\\
&\hat{N}|l_1l_2(L_{12})l_3:LM\rangle\\
&\quad=2(l_1+l_2+l_3)|l_1l_2(L_{12})l_3:LM\rangle.
\end{aligned} \tag{7·28}$$

In the above, the fact that the coupled angular momentum L_{12} is nothing but the isospin I is also recognized from the following relation:

$$\begin{aligned}
\hat{\boldsymbol{I}}^2 &= \frac{1}{2}(\hat{E}_2^1\hat{E}_1^2+\hat{E}_1^2\hat{E}_2^1)+\frac{1}{4}(\hat{E}_1^1-\hat{E}_2^2)^2\\
&=\frac{1}{2}\{(\hat{L}_{1+}+\hat{L}_{2+})(\hat{L}_{1-}+\hat{L}_{2-})+(\hat{L}_{1-}+\hat{L}_{2-})(\hat{L}_{1+}+\hat{L}_{2+})\}+(\hat{L}_{10}+\hat{L}_{20})^2.
\end{aligned} \tag{7·29}$$

In order to see the relation to the usual representation, we compare the action of $\hat{E}_1^3$ to the result obtained by K. T. Hecht.[8)]

$$\begin{aligned}
&\hat{E}_1^3|l_1l_2(L_{12})l_3:LM\rangle\\
&=\sum_{L'_{12}}(-)^{l_1+l_2+l_3+L+2L_{12}}\sqrt{(2L_{12}+1)(L'_{12}+1)2l_1(2l_1+1)(2l_3+1)(2l_3+2)}\\
&\quad\times\left\{\begin{array}{ccc} l_1 & l_2 & L_{12}\\ L'_{12} & \frac{1}{2} & l_1-\frac{1}{2}\end{array}\right\}\left\{\begin{array}{ccc} L_{12} & l_3 & L\\ l_3+\frac{1}{2} & L'_{12} & \frac{1}{2}\end{array}\right\}\left|l_1-\frac{1}{2}\,l_2(L'_{12})l_3+\frac{1}{2}:LM\right\rangle
\end{aligned}$$

$$= \sqrt{\frac{(L_{12}+l_1-l_2)(L_{12}+l_1+l_2+1)(L+L_{12}-l_3)(L-L_{12}+l_3+1)}{2L_{12}(2L_{12}+1)}}$$
$$\times \left| l_1 - \frac{1}{2}\, l_2 \left(L_{12} - \frac{1}{2} \right) l_3 + \frac{1}{2} : LM \right\rangle$$
$$- \sqrt{\frac{(l_2+L_{12}-l_1+1)(l_2-L_{12}+l_1)(L_{12}+l_3+L+2)(L_{12}+l_3-L+1)}{(2L_{12}+1)(2L_{12}+2)}}$$
$$\times \left| l_1 - \frac{1}{2}\, l_2 \left(L_{12} + \frac{1}{2} \right) l_3 + \frac{1}{2} : LM \right\rangle. \tag{7·30}$$

By the replacement

$$\begin{aligned} &L_{12} \longrightarrow I = \tfrac{1}{2}\lambda - \tfrac{1}{2}(p-q), \quad && l_1 - l_2 \longrightarrow I_0, \\ &l_3 \longrightarrow \tfrac{1}{2}w = \tfrac{1}{2}(p+q), && l_1 + l_2 + l_3 \longrightarrow \mu + \tfrac{1}{2}\lambda, \\ &L \longrightarrow \tfrac{1}{2}\lambda, \end{aligned} \tag{7·31}$$

the action of $\hat{E}_1^3$ can be rewritten in the form

$$\begin{aligned} &\hat{E}_1^3 | l_1 l_2 (L_{12}) l_3 : LM \rangle \\ &= \sqrt{\frac{(I+I_0)(\lambda+\mu+1-p)(\lambda-p)(p+1)}{2I(2I+1)}} \left| l_1 - \frac{1}{2}\, l_2 \left(I - \frac{1}{2} \right) l_3 + \frac{1}{2} : LM \right\rangle \\ &\quad - \sqrt{\frac{(I-I_0+1)(\mu-q)(\lambda+q+2)(q+1)}{(2I+1)(2I+2)}} \left| l_1 - \frac{1}{2}\, l_2 \left(I + \frac{1}{2} \right) l_3 + \frac{1}{2} : LM \right\rangle \\ &= -\left(\frac{1}{2} - \frac{1}{2}\, I I_0 \middle| I - \frac{1}{2}\, I_0 - \frac{1}{2} \right) \sqrt{\frac{(\lambda+\mu+1-p)(\lambda-p)(p+1)}{2I}} \\ &\qquad \times \left| l_1 - \frac{1}{2}\, l_2 \left(I - \frac{1}{2} \right) l_3 + \frac{1}{2} : LM \right\rangle \\ &\quad - \left(\frac{1}{2} - \frac{1}{2}\, I I_0 \middle| I + \frac{1}{2}\, I_0 - \frac{1}{2} \right) \sqrt{\frac{(\mu-q)(\lambda+q+2)(q+1)}{(2I+2)}} \\ &\qquad \times \left| l_1 - \frac{1}{2}\, l_2 \left(I + \frac{1}{2} \right) l_3 + \frac{1}{2} : LM \right\rangle. \end{aligned} \tag{7·32}$$

This expression is nothing but the result obtained in Ref. 8). The quantum numbers in the present representation have an apparent correspondence to those in the usual representation. The quantum number M denotes the eigenvalue of $\frac{1}{2}(\hat{F}_1^1 - \hat{F}_2^2)$, which is commutable with every operators $\hat{E}_q^p$. This value is specified as the initial condition where the fermions are distributed arbitrarily to the first and the second single-particle levels (in the quark model, the number of u and d quarks are freely fixed). Then we can derive the irreducible representation available under the initial distribution.

At the end of this subsection, we comment on the broken symmetry of $su(3)$. Let us take up the operator

$$\hat{L}_{30} = \frac{1}{2}(\hat{x}_{13}^\dagger \hat{x}_{13} - \hat{x}_{23}^\dagger \hat{x}_{23}). \tag{7·33}$$

This operator does not commute with the quasi-spin of the F-space:

$$[\hat{\boldsymbol{L}}^2, \hat{L}_{30}]_- = \frac{1}{2}(\hat{L}_+ L_{3-} - \hat{L}_{3+}\hat{L}_- - \hat{L}_-\hat{L}_{3+} + \hat{L}_{3-}\hat{L}_+)$$
$$= \sum_{a=1}^{2}(\hat{L}_{a+}\hat{L}_{3-} - \hat{L}_{3+}\hat{L}_{a-}) \neq 0. \tag{7·34}$$

When the hamiltonian involves such an operator, therefore, there occurs a mixing among different values of λ, as shown in the following:

$$\hat{L}_{30}|l_1 l_2(L_{12})l_3 : LM\rangle$$
$$= \sum_{L'}(-)^{L_{12}+l_3+M+1}\sqrt{l_3(l_3+1)(2l_3+1)(2L+1)(2L'+1)}$$
$$\times \begin{pmatrix} 1 & L & L' \\ 0 & M & -M \end{pmatrix} \begin{Bmatrix} 1 & L & L' \\ L_{12} & l_3 & l_3 \end{Bmatrix} |l_1 l_2(L_{12})l_3 : L'M\rangle, \tag{7·35}$$

where $(***)$ denotes the 3-j symbol related to the Clebsch-Gordan coefficient through by the relation

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} = \frac{(-)^{j_1-j_2+m_3}}{\sqrt{2j_3+1}}(j_1 m_1 j_2 m_2 | j_3 m_3). \tag{7·36}$$

The relation (7·35) is derived in the following way:

$$\hat{L}_{30}|l_1 l_2(L_{12})l_3 : LM\rangle$$
$$= \sum_{M_{12},m_3} m_3(L_{12}M_{12}l_3m_3|LM)|l_1 l_2 : L_{12}M_{12}\rangle \otimes |l_3 m_3\rangle$$
$$= \sum_{M_{12},m_3,L'} m_3(L_{12}M_{12}l_3m_3|LM)(L_{12}M_{12}l_3m_3|L'M)|l_1 l_2(L_{12})l_3 : L'M\rangle,$$
$$\sum_{M_{12},m_3} m_3(L_{12}M_{12}l_3m_3|LM)(L_{12}M_{12}l_3m_3|L'M)$$
$$= \sum_{M_{12},m_3}(-)^{l_3-m_3}\sqrt{l_3(l_3+1)(2l_3+1)(2L+1)(2L'+1)}$$
$$\times \begin{pmatrix} l_3 & l_3 & 1 \\ m_3 & -m_3 & 0 \end{pmatrix} \begin{pmatrix} L_{12} & l_3 & L \\ M_{12} & m_3 & -M \end{pmatrix} \begin{pmatrix} L_{12} & l_3 & L' \\ M_{12} & m_3 & -M \end{pmatrix}$$
$$= \sum_{M_{12},m_3}(-)^{l_3-m_3}(-)^{2l_3+1}\sqrt{l_3(l_3+1)(2l_3+1)(2L+1)(2L'+1)}$$
$$\times \begin{pmatrix} 1 & l_3 & l_3 \\ 0 & -m_3 & m_3 \end{pmatrix} \begin{pmatrix} L_{12} & L & l_3 \\ -M_{12} & M & -m_3 \end{pmatrix} \begin{pmatrix} L_{12} & l_3 & L' \\ M_{12} & m_3 & -M \end{pmatrix}$$
$$= (-)^{L_{12}+l_3+M+1}\sqrt{l_3(l_3+1)(2l_3+1)(2L+1)(2L'+1)}$$
$$\times \begin{pmatrix} 1 & L & L' \\ 0 & M & -M \end{pmatrix} \begin{Bmatrix} 1 & L & L' \\ L_{12} & l_3 & l_3 \end{Bmatrix}.$$

7.2. *Retracted $u(1)$ algebra*

In the case $n_2 = n_3 = 0$, the F-$u(3)$ algebra is retracted to a $u(1)$ algebra:

$$\hat{E}^p_q = \hat{x}^\dagger_{1p}\hat{x}_{1q}; \quad p, q = 1, 2, 3, \tag{7·37a}$$

$$\hat{F}^a_b = \begin{cases} \sum_{p=1}^3 \hat{x}^\dagger_{1p}\hat{x}_{1p}; & a = b = 1, \\ 0 \quad ; & \text{otherwise.} \end{cases} \tag{7·37b}$$

The basis vector of the irreducible representation carrier space can be specified by the set of quantities (N_1, N_2, N_3):

$$|N_1N_2N_3\rangle = \prod_{p=1}^3 \frac{1}{\sqrt{N_p!}}(\hat{x}^\dagger_{1p})^{N_p}|0\rangle_x. \tag{7·38}$$

This state just corresponds to the state $|\prod_{p=1}^3[\frac{N_p}{2}\frac{N_p}{2}]\rangle$, i.e.,

$$|N_1N_2N_3\rangle = \left|\frac{N_1}{2}\frac{N_2}{2}\left(\frac{N_1+N_2}{2}\right)\frac{N_3}{2} : \frac{N}{2}\frac{N}{2}\right\rangle. \quad \left(N = \sum_{p=1}^3 N_p\right) \tag{7·39}$$

In this case, therefore, we take up the subspace of the irreducible representation carrier space specified by

$$\begin{aligned} &L_{12} = \tfrac{N_1+N_2}{2} \to I = \tfrac{1}{2}\lambda - \tfrac{1}{2}(p-q), \quad && l_1 - l_2 = \tfrac{N_1-N_2}{2} \to I_0, \\ &l_3 = \tfrac{N_3}{2} \to \tfrac{1}{2}w = \tfrac{1}{2}(p+q), && l_1 + l_2 + l_3 = \tfrac{N_1+N_2+N_3}{2} \to \mu + \tfrac{1}{2}\lambda, \\ &L = \tfrac{N_1+N_2+N_3}{2} \to \tfrac{1}{2}\lambda, \end{aligned}$$

that is,

$$\begin{gathered} \lambda = N_1 + N_2 + N_3, \quad \mu = 0, \\ Y = \frac{1}{3}(N_1 + N_2 - 2N_3), \quad I = \frac{N_1+N_2}{2}, \quad I_0 = \frac{N_1-N_2}{2}. \\ (p = N_3, \quad q = 0) \end{gathered} \tag{7·40}$$

In this restricted subspace, the action of $\hat{E}^p_q$ becomes very simple, for instance,

$$\begin{aligned} &\hat{E}^p_p\left|\frac{N_1}{2}\frac{N_2}{2}\frac{N_3}{2}\right\rangle = \frac{N_p}{2}\left|\frac{N_1}{2}\frac{N_2}{2}\frac{N_3}{2}\right\rangle, \\ &\hat{E}^1_2\left|\frac{N_1}{2}\frac{N_2}{2}\frac{N_3}{2}\right\rangle = \sqrt{(N_1+1)N_2}\left|\frac{N_1+1}{2}\frac{N_2-1}{2}\frac{N_3}{2}\right\rangle, \\ &\hat{E}^2_1\left|\frac{N_1}{2}\frac{N_2}{2}\frac{N_3}{2}\right\rangle = \sqrt{N_1(N_2+1)}\left|\frac{N_1-1}{2}\frac{N_2+1}{2}\frac{N_3}{2}\right\rangle. \end{aligned} \tag{7·41}$$

The action of other $\hat{E}^p_q$ can be also written in a similar form.

In this restricted subspace, $\hat{E}^p_q$ can be expressed in the following form:

$$\begin{aligned} &\hat{E}^1_1 = \hat{N} - \textstyle\sum_{p=2}^3 \hat{X}^\dagger_{1p}\hat{X}_{1p}, \\ &\hat{E}^1_p = \sqrt{\hat{N} - \textstyle\sum_{r=2}^3 \hat{X}^\dagger_{1r}\hat{X}_{1r}}\,\hat{X}_{1p}, \quad && \hat{E}^p_1 = \hat{X}^\dagger_{1p}\sqrt{\hat{N} - \textstyle\sum_{r=2}^3 \hat{X}^\dagger_{1r}\hat{X}_{1r}}; \quad p \neq 1, \\ &\hat{E}^p_q = \hat{X}^\dagger_{1p}\hat{X}_{1q}; && p, q = 2, 3, \end{aligned} \tag{7·42}$$

and the basis vector of the restricted irreducible representation carrier space can be expressed as

$$|N_1N_2N_3\rangle = \exp[i(N_1+N_2+N_3)\hat{\phi}]\frac{1}{\sqrt{N_2!N_3!}}(\hat{X}_{12}^{\dagger})^{N_2}(\hat{X}_{13}^{\dagger})^{N_3}|0\rangle_X. \tag{7·43}$$

In the above, the new variables satisfy the commutation relations

$$\begin{gathered}[\hat{X}_{1p},\hat{X}_{1q}^{\dagger}]_- = \delta_{pq}, \quad [\hat{X}_{1p},\hat{X}_{1q}]_- = 0,\\ [\hat{N},\exp[i\hat{\phi}]]_- = \exp[i\hat{\phi}],\\ [\hat{N},\hat{X}_{1p}]_- = [\exp[i\hat{\phi}],\hat{X}_{1p}]_- = 0.\end{gathered} \tag{7·44}$$

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Appendix A

—— The Action of $\hat{E}_q^p$ of u(3) Algebra ——

Here we derive the action of $\hat{E}_3^1$.

$$\begin{aligned}
&\hat{E}_3^1|l_1l_2(L_{12})l_3:LM\rangle\\
&=\sum_{L_{13}}(-)^{l_2+l_3+L_{12}+L_{13}}\sqrt{(2L_{12}+1)(2L_{13}+1)}\\
&\quad\times\left\{\begin{array}{ccc} l_1 & l_2 & L_{12}\\ L & l_3 & L_{12}\end{array}\right\}\hat{E}_3^1|l_1l_3(L_{13})l_2:LM\rangle\\
&=\sum_{L_{13}}(-)^{l_2+l_3+L_{12}+L_{13}}\sqrt{(2L_{12}+1)(2L_{13}+1)(L_{13}-l_1+l_3)(L_{13}+l_1-l_3+1)}\\
&\qquad\times\left\{\begin{array}{ccc} l_1 & l_2 & L_{12}\\ L & l_3 & L_{13}\end{array}\right\}\left|l_1+\frac{1}{2}\,l_3-\frac{1}{2}(L_{13})l_2:LM\right\rangle\\
&=\sum_{L_{13}}(-)^{l_2+l_3+L_{12}+L_{13}}\sqrt{(2L_{12}+1)(2L_{13}+1)(L_{13}-l_1+l_3)(L_{13}+l_1-l_3+1)}\\
&\quad\times\left\{\begin{array}{ccc} l_1 & l_2 & L_{12}\\ L & l_3 & L_{13}\end{array}\right\}\sum_{L'_{12}}(-)^{l_2+l_3+L_{13}+L'_{12}-\frac{1}{2}}\sqrt{(2L'_{12}+1)(2L_{13}+1)}\\
&\qquad\times\left\{\begin{array}{ccc} l_1+\frac{1}{2} & l_3-\frac{1}{2} & L_{13}\\ L & l_2 & L'_{12}\end{array}\right\}\left|l_1+\frac{1}{2}\,l_2(L'_{12})l_3-\frac{1}{2}:LM\right\rangle.
\end{aligned} \tag{A·1}$$

With the aid of the formulae

$$\sqrt{(L_{13}-l_1+l_3)(L_{13}+l_1-l_3+1)}$$
$$=(-)^{l_1+l_3+L_{13}}\sqrt{2l_3(2l_3+1)(2l_1+1)(2l_1+2)}\left\{\begin{array}{ccc} L_{13} & l_1 & l_3 \\ \frac{1}{2} & l_3-\frac{1}{2} & l_1+\frac{1}{2}\end{array}\right\}, \quad \text{(A·2)}$$

$$\sum_{L_{13}}(-)^{L_{13}}(2L_{13}+1)\left\{\begin{array}{ccc} l_1 & l_3 & L_{13} \\ L & l_2 & L_{12}\end{array}\right\}\left\{\begin{array}{ccc} l_1+\frac{1}{2} & l_3-\frac{1}{2} & L_{13} \\ l_3 & l_1 & \frac{1}{2}\end{array}\right\}$$
$$\times\left\{\begin{array}{ccc} l_1+\frac{1}{2} & l_3-\frac{1}{2} & L_{13} \\ L & l_2 & L'_{12}\end{array}\right\}$$
$$=(-)^{l_2+L_{12}+L'_{12}+L+\frac{1}{2}}\left\{\begin{array}{ccc} l_1 & l_2 & L_{12} \\ L'_{12} & \frac{1}{2} & l_1+\frac{1}{2}\end{array}\right\}\left\{\begin{array}{ccc} l_3 & L & L_{12} \\ L'_{12} & \frac{1}{2} & l_3-\frac{1}{2}\end{array}\right\}, \quad \text{(A·3)}$$

we have the relation

$$\hat{E}^1_3|l_1l_2(L_{12})l_3:LM\rangle$$
$$=\sum_{L'_{12}}(-)^{l_1+l_2+l_3+2L'_{12}+L}\sqrt{(2L_{12}+1)(2L'_{12}+1)2l_3(2l_3+1)(2l_1+1)(2l_1+2)}$$
$$\times\left\{\begin{array}{ccc} l_1 & l_2 & L_{12} \\ L'_{12} & \frac{1}{2} & l_1+\frac{1}{2}\end{array}\right\}\left\{\begin{array}{ccc} l_3 & L & L_{12} \\ L'_{12} & \frac{1}{2} & l_3-\frac{1}{2}\end{array}\right\}\left|l_1+\frac{1}{2}\,l_2(L'_{12})l_3-\frac{1}{2}:LM\right\rangle. \quad \text{(A·4)}$$

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