

Exact Ground-State Energies of the Random-Field Ising Chain and Ladder

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We derive the exact ground-state energies of the random-field Ising chain and ladder. The random fields consist of three possible values for the chain model and bimodal fields for the ladder model. The results are piecewise linear as a function of the random-field strength. The zero-temperature transfer matrix method is used to obtain these results.

§1. Introduction

The one-dimensional random Ising model is one of the simplest examples of systems with quenched disorder. In the long history of its intensive studies, a number of results have been published which give exact solutions in the ground state.^{1)–5)} Mattis and Paul⁶⁾ have analyzed the random-bond Ising model on the ladder geometry at finite temperature. Their result does not have a closed analytical form because the probability distribution of eigenvalue involves devil's staircase. Kadowaki et al.⁵⁾ calculated explicitly the exact ground-state energies of the $\pm J$ random bond model and the site-random model on strips of various widths. To obtain these exact results, they used the transfer matrix method at zero-temperature. Being motivated by the method proposed in Ref. 5), we apply their method to the Ising chain in three possible random fields and Ising ladder in $\pm h$ random fields.⁷⁾ The solutions are both piecewise linear as a function of the random-field strength. In the next section, we consider the Ising chain in random fields taking $\pm h$ and 0 and in §3 we show the exact ground-state energy of the Ising ladder in $\pm h$ random fields.

§2. Random-field Ising chain

2.1. Recursion relations

First, we consider the Ising chain in three possible random fields $\pm h$ and 0. The random field is chosen to be $+h$ with probability p_1 , 0 with p_2 and $-h$ with p_3 and these probabilities satisfy $p_1 + p_2 + p_3 = 1$. Considering the transfer matrix at zero-temperature, we obtain the recursion relations,⁵⁾

$$x_{n+1} = x_n + J + h_{n+1} + 2 \max(0, a_n - J), \quad (2.1)$$

$$a_{n+1} = f(a_n) - h_{n+1}, \quad (2.2)$$

where x_n is the ground-state energy with the n 'th spin state being $+1$, J is the ferromagnetic coupling constant and h_{n+1} is the random field at site $n + 1$. The energy difference between the edge spin state being $+1$ and -1 is represented by a_n . The function $f(x)$ is piecewise linear as $+J$ for $x > J$, $-J$ for $x < -J$ and x for $-J < x < J$. As shown below, using the recursive calculation of the above two

relations, the average of the ground state energy over random fields reduces to the average of $\max(0, a_n - J)$ over the distribution of the energy difference a_n .

Using the relation (2.1), we can express the ground-state energy per spin as⁵⁾

$$E_{\text{GS}} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} (x_{n+1} - x_n) = -J - (p_1 - p_3)h - 2 \langle \langle \max(0, a - J) \rangle \rangle, \quad (2.3)$$

where a is the energy difference in the limit $N \rightarrow \infty$ and the double brackets $\langle \langle \dots \rangle \rangle$ mean the average over the probability distribution of the energy difference $P(a)$. To obtain the last expression (2.3), we have invoked the self-averaging property of the energy.

2.2. Probability distribution

For explicit evaluation of the ground-state energy, we should find the probability distribution of the energy difference $P(a)$ according to Eq. (2.2). Here we briefly indicate how to evaluate the probability distribution $P(a)$ from the recursion relation (2.2). First we choose an arbitrary initial condition. Next we iterate this recursion relation and obtain a finite set of $\{a\}$ as a stationary state of the recursion relation. This statistical weight of the stationary state is obtained by solving the eigenvalue problem of transition matrix which expresses the invariant process under Eq. (2.2).

Before we show the detailed expression of probability distribution $P(a)$, we explain another important feature of the ground state of the present model. The ground-state spin configurations are invariant in the range

$$\frac{2}{k+1}J < h < \frac{2}{k}J. \quad (2.4)$$

Within this range the ground-state energy is a linear function of the random-field strength h . This feature is understood by a simple argument of the ground-state spin configuration. Let us consider several clusters of fields taking $-h$ in the sea of fields taking $+h$. For $h > 2J$, all spins are parallel to the random fields. When h decreases and satisfies $h < 2J$, a single spin flips its direction from down to up. For smaller field $h < J$, a cluster of two neighboring spins flips their direction from down to up. The ground state proceeds with this change, and k -spin cluster flips its direction from $-h$ to $+h$ when the random-field strength reaches $h = 2J/k$. Therefore, the ground-state spin configuration is invariant in the range (2.4).

From Eq. (2.4) and the evaluation method of the probability distribution $P(a)$ suggested above, we obtain the following probability distribution $P_k(a)$ within the k 'th range of the random field (2.4):

$$P_k(a(i, j, m)) = p_j D_m^i, \quad (2.5)$$

where

$$D_m^1 = \frac{\sum_{l=m}^{k+1} \left(\frac{p_3}{p_1}\right)^{l-1}}{\sum_{l=1}^{k+1} \left(\frac{p_3}{p_1}\right)^{l-1} \sum_{l=1}^{k+2} \left(\frac{p_3}{p_1}\right)^{l-1}}, \quad D_m^2 = \frac{\sum_{l=m}^{k+1} \left(\frac{p_1}{p_3}\right)^{l-1}}{\sum_{l=1}^{k+1} \left(\frac{p_1}{p_3}\right)^{l-1} \sum_{l=1}^{k+2} \left(\frac{p_1}{p_3}\right)^{l-1}}, \quad (2.6)$$

and $a(i, j, m) = (-1)^i(J - (m - 1)h) - (j - 2)h$. Here i, j and m are integers, and i is chosen from 1 or 2, j runs from 1 to 3 and m runs from 1 to $k + 1$. The stationary state in the k 'th region consists of $2 \times 3 \times (k + 1)$ possible points.

Using the probability distribution $P_k(a(i, j, m))$, we finally obtain the ground-state energy for $2J/(k + 1) < h < 2J/k$ as

$$E_{\text{GS}} = \left\{ -1 + \frac{4(p_1 p_3)^{k+1} (p_1 - p_3)^2}{A(k+1)A(k+2)} \right\} J - \left\{ 1 - p_2 + \frac{2(p_1 p_3)^{k+1} (k+1) (p_1 - p_3)^2}{A(k+1)A(k+2)} - \frac{2A(k+1)p_1 p_3}{A(k+2)} \right\} h, \quad (2.7)$$

where $A(k)$ is

$$A(k) = p_1^k - p_3^k. \quad (2.8)$$

§3. Random-field Ising ladder

Next, we consider the Ising spin ladder in bimodal random fields. For this model, we can prove its equivalence to the Ising chain discussed in the previous section when $h < J$. Let us consider the specific random-field configuration which consists of the fields being $-h$ in the first chain and $+h$ in the second chain. For this random-field configuration, there are two possible ground states. One consists of all spins being up (or down due to the symmetry with energy E_{allup}), and the other is all spins being parallel to the field (energy E_{fields}). Their energies are

$$E_{\text{allup}} = -2(n - 1)J - nh + n(h - J), \quad (3.1)$$

$$E_{\text{fields}} = -2(n - 1)J - nh + n(J - h), \quad (3.2)$$

where n is the number of the rungs of the ladder. If the random-field strength h is smaller than J , the ground-state energy becomes E_{allup} , and the ground state is all spins being up or down with equal probability. Any random-field configuration can be divided into the cluster of the random field consisting of the fields being $-h$ in the first chain and $+h$ in the second chain. Consequently, we conclude that for $h < J$, two spins on the same rung always point to the same direction. Therefore, by identifying spins on the same rung as a spin taking the values ± 1 , we obtain the effective coupling constant $2J$ and the three kinds of effective fields $\{+2h, 0, -2h\}$.

The exact ground-state energy of the ladder model can therefore be obtained from Eq. (2.7). The corresponding probabilities of the random field are $p_1 = x^2$ for $+h$, $p_2 = 2x(1 - x)$ for 0 and $p_3 = (1 - x)^2$ for $-h$ where x means the probability of the random field being $+h$ for the ladder model. We also replace the random field strength and the coupling constant of the one-dimensional model with $J \rightarrow 2J$ and $h \rightarrow 2h$. We need an additional term J which is caused by spins on the same rung pointing to the same direction, and divide the result by 2 because the number of spins in ladder model is twice as many as the one-dimensional model. Consequently, the final expression is obtained from Eq. (2.7) by replacing the coefficient -1 to $-3/2$.

§4. Discussion

We give a few comments on some aspects other than the ground-state energy. First, it should be possible to derive the exact solution for other physical quantities than the ground-state energy, such as the entropy and magnetization, using the methods in Refs. 1) and 8). Investigations in this direction are under way.

To increase the number of the legs in the ladder model is quite interesting toward the analysis of the two-dimensional system. Actually, we have found that the M -leg random-field Ising ladder can be mapped to the chain model with $M + 1$ possible random fields for $h < 2J/M$. It is necessary to extend this range of validity since $2J/M$ vanishes in the two-dimensional limit $M \rightarrow \infty$. However, for the Ising ladder in bimodal random fields, the evaluation of the transition matrix that lead to (2·1) and (2·2) is not straightforward for $h > J$ since we cannot map the problem to the one-dimensional model. This difficulty is caused by the complicated structure of the recursion relation of the ladder model. In fact, we find four kinds of recursion relations to evaluate the probability distribution of the energy differences. It is not easy to construct the general form of the transition matrix from these recursion relations. This is nevertheless an important and challenging task because of the eventual target of the two-dimensional model as mentioned above.

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References

- 1) B. Derrida, J. Vannimenus and Y. Pomeau, *J. of Phys. C* **11** (1978), 4749.
- 2) E. Farhi and S. Gutmann, *Phys. Rev. B* **48** (1993), 9508.
- 3) A. Vilenkin, *Phys. Rev. B* **18** (1978), 1474.
- 4) C. Dress, E. Amic and J. M. Luck, *J. of Phys. A* **28** (1995), 135.
- 5) T. Kadowaki, Y. Nonomura and H. Nishimori, *J. Phys. Soc. Jpn.* **65** (1996), 1609.
- 6) D. C. Mattis and P. Paul, *Phys. Rev. Lett.* **83** (1999), 3733.
- 7) T. Hamasaki and H. Nishimori, *J. Phys. Soc. Jpn.* **73** (2004), 1490.
- 8) P. Rujan, *Phys. Rev. Lett.* **70** (1993), 2968.