

Statistical Mechanics of Resource Allocation

Andrea DE MARTINO

*INFN-SMC and Dipartimento di Fisica, Università di Roma "La Sapienza",
p. le A. Moro 2, 00185 Roma, Italy*

We sketch the usefulness of spin-glass techniques for the analysis of the macroscopic properties of certain resource allocation problems, focusing in particular on a simple linear production model and on the competitive equilibrium model.

The efficiency with which resources are allocated is a central issue in economics. The concept of ‘allocation’ is to be intended here in a broad sense that includes both the exchange of commodities among agents and the production of commodities by means of commodities whereas ‘efficiency’ is usually connected to the solutions of maximum or minimum problems, as for instance firms try to meet demands at minimum costs.¹⁾ A question that naturally arises concerns the collective properties of efficient states, or the aggregation of the microeconomic behavior of producers, consumers, etc. into macroeconomic laws — like the distribution of prices, consumption levels or firm sizes — that can eventually be checked against empirical observations. To this aim, it is important to take into account the existence of heterogeneities across agents, that is of different budgets, endowments, technologies and preferences. Traditional economic methods like the so-called representative-agent approach are inadequate in this respect.²⁾ Mean-field spin-glass theory offers an alternative toolbox whose usefulness in many cases goes beyond the merely technical level.

To begin with, we consider the classical linear model of production to meet demand at minimum cost. Let there be N processes (or technologies) labeled by i and P commodities labeled by μ . Each process allows the transformation of some commodities (inputs) into others (outputs) and is characterized by an input-output vector $\xi_i = \{\xi_i^\mu\}$ where negative (positive) components represent quantities of inputs (outputs). Moreover it can be operated at any scale $s_i \geq 0$ and the cost of operating it at scale $s_i = 1$ is p_i . One wants to choose the s_i 's so as to minimize the total cost $\sum_i s_i p_i$ subject to the requirements that $\sum_i s_i \xi_i^\mu = \kappa^\mu$ for all μ , so that if $\kappa^\mu > 0$ ($\kappa^\mu < 0$) the total amount of commodity μ that is produced (consumed) matches a fixed demand (availability). A simple question to ask is how the operation pattern changes when N increases, i.e. as more technologies become available. Indeed, the macroscopic structure of the efficient state must be expected to depend on the ratio N/P : for $N \ll P$ a technology will be more likely to be active ($s_i > 0$) than for $N \gg P$, when selection will be stronger and processes performing the required conversions more efficiently will be favored. In the limit $N \rightarrow \infty$ with $n = \lim_{N \rightarrow \infty} N/P$ fixed one can resort to spin-glass techniques after assuming that the ξ_i^μ 's are independent identically distributed quenched Gaussian random variables with zero mean and variance $1/P$ (which makes the model similar to the knapsack problem³⁾). In order

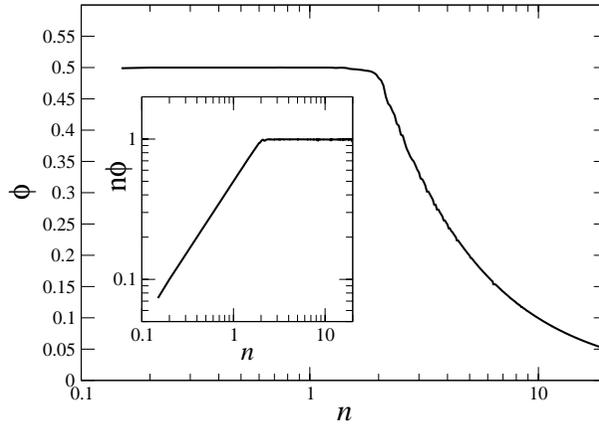


Fig. 1. Fraction ϕ of active processes vs n for $\eta = 0.001$, $p_i = 1$ and $m = 0.1$. Inset: ϕn vs n for the same parameter values.

to obtain non-trivial results it is necessary to impose also that $\sum_{\mu} \xi_i^{\mu} = -\eta$ with $\eta > 0$ for all i . This ensures that it is impossible to produce, by combining processes, a finite amount of some commodity without consuming a finite amount of some other commodity. Also, we assume that the κ^{μ} 's and p_i 's are quenched random variables. The solutions are described by the quantity $f = \lim_{N \rightarrow \infty} \frac{1}{\beta N} \langle \langle \log Z \rangle \rangle_{\xi}$ with 'partition function'

$$Z = \int_0^{\infty} ds e^{-\beta \sum_i s_i p_i} \prod_{\mu} \delta \left(\sum_i s_i \xi_i^{\mu} - \kappa^{\mu} \right) \quad (1)$$

in the limit $\beta \rightarrow \infty$. Here $\langle \langle \dots \rangle \rangle_{\xi} = \langle \dots \prod_i \delta(\sum_{\mu} \xi_i^{\mu} + \eta) \rangle_{\xi} / \langle \prod_i \delta(\sum_{\mu} \xi_i^{\mu} + \eta) \rangle_{\xi}$ and $\langle \dots \rangle_{\xi}$ stands for an average over ξ_i 's. The evaluation of f requires a standard replica calculation. As usual, after carrying out the disorder average, factorizing over technologies and commodities, and performing a saddle-point integration one obtains an 'effective-spin' problem that in this case can be interpreted as cost minimization by a representative technology. Its solution s^* , which depends also on the order parameters generated by the disorder average, can be interpreted as the optimal scale of production. Starting from s^* it is possible to evaluate all interesting macroscopic observables such as the distribution of operation scales and the probability that a randomly drawn technology will be active as a function of n . The latter quantity is displayed in Fig. 1 in the case in which $p_i = 1$ for all i while thresholds are distributed according to a simple binomial $q(\kappa) = \frac{1-m}{2} \delta(\kappa + 1) + \frac{1+m}{2} \delta(\kappa - 1)$ (a more thorough account of the model's properties will be given elsewhere). One sees that for small n roughly a half of the processes are active. This means that as n increases, that is as more and more technologies become available, the number of active processes per good increases (see inset) i.e. the arrival of new technologies favors existing ones. The picture changes radically for $n \gtrsim 2$, as ϕ starts to decrease and $\phi n = 1$. Now the number of active processes equals that of commodities and technologies undergo a much stronger selection which reduces the probability that a randomly drawn input-output vector is active. This describes in a nutshell a transition to a highly competitive state where all possible productions are saturated by existing

technologies and an increase in activity levels can be achieved only by increasing P .

In general demands and production costs are not fixed and the coordination of supply and demand is guided by prices. The basic model of this type is the competitive equilibrium model.^{1),4)} One considers, as before, P commodities and N technologies. The profit each of them faces when operating at scale s_i is $\pi_i = s_i(\boldsymbol{\xi}_i \cdot \mathbf{p})$, where $\mathbf{p} = \{p^\mu\}$ is the vector of prices with $p^\mu \geq 0$. Each firm aims at finding the s_i that maximizes π_i at fixed prices. The consumer ('society') instead chooses his consumption $\mathbf{x} = \{x^\mu\}$ ($x^\mu \geq 0$) so as to maximize a given utility function $U(\mathbf{x})$ subject to the requirement that $\mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{y}$, where $\mathbf{y} = \{y^\mu\}$ ($y^\mu \geq 0$) is the initial endowment of goods in the society. The optimal states ('equilibria') are those for which the above problems are simultaneously solved and the total demand of each commodity equals the total supply: $\mathbf{x} - \mathbf{y} = \sum_i s_i \boldsymbol{\xi}_i$. This condition ('market clearing') ultimately determines the optimal prices. The importance of this scheme lies not so much in realism, but rather in its ability to account for the interplay between competition, price formation and allocative efficiency in economic systems (see Ref. 1) for a more detailed economic discussion). Statistical mechanics allows the evaluation of optimal quantities in the limit $N \rightarrow \infty$ as a function of $n = \lim_{N \rightarrow \infty} N/P$ when input-output vectors and initial endowments are quenched random variables. This analysis, whose starting point is again the 'free energy' f with

$$Z = \int_0^\infty d\mathbf{x} e^{\beta U(\mathbf{x})} \int_0^\infty ds \delta\left(\mathbf{x} - \mathbf{y} - \sum_i s_i \boldsymbol{\xi}_i\right) \quad (2)$$

in place of (1) (it can be shown that $\pi_i = 0$ for all i at equilibrium), has been carried out in Ref. 5) under the assumption that $U(\mathbf{x}) = \sum_\mu u(x^\mu)$ (separability) with $u'(x) > 0$ and $u''(x) < 0$. Separability implies that commodities are *a priori* equivalent. Hence this model focuses on the ability of the economy to produce scarce goods (namely goods with y^μ small) using abundant goods (namely goods with y^μ large) as inputs and one aims at understanding how the collective prosperity level, measured by the amount of productive activity and by the utility level of the society, changes as the repertoire of available technologies expands, that is as n increases.

As before, the replica approach leads to effective-agent problems, namely profit maximization by an effective firm and utility maximization with respect to an effective commodity, from which the equilibrium operation scale and consumption are easily derived. Remarkably at equilibrium prices and price fluctuations, which do not appear explicitly in (2), turn out to be directly connected to the spin-glass order parameters generated by the disorder averaging. We refer the reader to Ref. 5) for technicalities and just review the final outcome for the case in which initial endowments y^μ are exponentially distributed, $\rho(y) = e^{-y}$ and $u(x) = \log x$ (see Fig. 2). For small n , the optimal operation scale s^* increases with increasing n , signaling that the introduction of new technologies favors existing ones. At the same time, the optimal consumption and the relative consumption fluctuations decrease, as firms are managing the transformation of abundant goods into scarce ones. This 'expanding' picture changes drastically at $n_c \simeq 2$, where the relative consumption fluctuations drop. Now it becomes harder and harder to identify abundant goods and the econ-

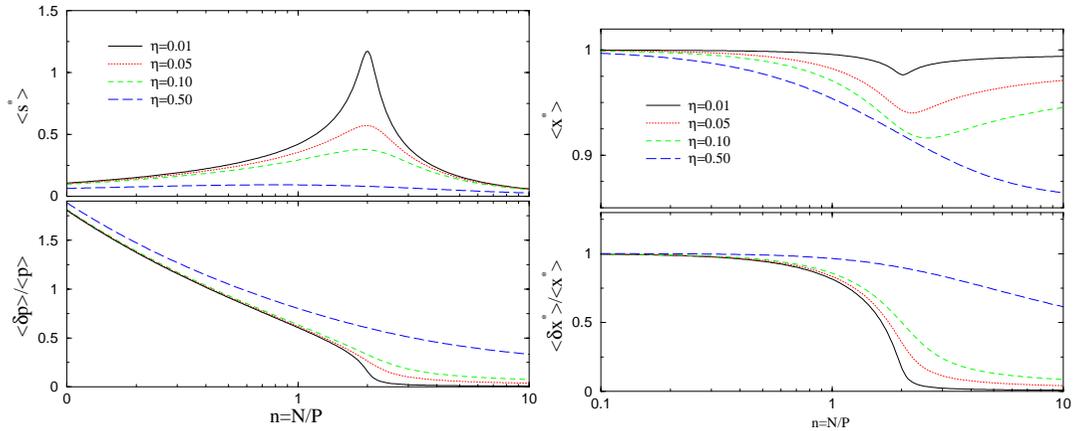


Fig. 2. Left: typical equilibrium scale of production (top) and relative price fluctuations (bottom). Right: typical equilibrium consumption (top) and relative consumption fluctuations (bottom).

omy becomes strongly selective towards more efficient technologies, as signaled by the fact that s^* decreases as n increases. In parallel, consumptions increase as higher utility levels can now only be achieved in this way. Loosely speaking: the economy has entered a mature phase. It can be shown that $n_c = 2$ is the critical point of a second-order phase transition for $\eta \rightarrow 0$. Furthermore, if one allows for an extra degree of variability in the technology distribution by imposing that the variance of ξ_i^μ equals Δ_i/P with Δ_i independent, identically distributed quenched random variables with probability density $g(\Delta)$, one obtains for the distribution of activity levels a scaling form $\mathcal{P}(s) \propto s^{-3-2\ell}$ whenever $g(\Delta) \propto \Delta^\ell$ for $\Delta \ll 1$. A power-law distribution has been found in many empirical studies of firm sizes (see for example Ref. 6)). Again, the reader is referred to Ref. 5) for a more detailed analysis.

In summary, we have briefly addressed the usefulness of spin-glass techniques for the analysis of certain economically-inspired optimization problems. Of the many open directions for improvements, we would like to emphasize the extension of these results to economies with many heterogeneous consumers and to finite-connectivity contexts in which technologies consume and produce a finite number of commodities. Work along these lines is in progress.

It is my pleasure to thank M. Marsili and I. Perez Castillo for a fruitful and enjoyable collaboration, as well as I. Giardina, F. Krzakala, E. Marinari, A. Tedeschi and M. A. Virasoro for countless discussions and suggestions.

References

- 1) A. Mas-Colell, M. D. Whinston and J. R. Green, *Microeconomic theory* (Oxford University Press, Oxford, 1995).
- 2) A. P. Kirman, *J. Econ. Persp.* **6** (1992), 117.
- 3) E. Korutcheva, M. Opper and B. Lopez, *J. of Phys. A* **27** (1994), L645.
J. Inoue, *J. of Phys.* **30** (1997), 1047.
- 4) K. J. Lancaster, *Mathematical economics* (Dover, New York, NY, 1987).
- 5) A. De Martino, M. Marsili and I. Perez Castillo, cond-mat/0309533.
A. De Martino, M. Marsili and I. Perez-Castillo, *J. Stat. Mech.* **1** (2004), P04002.
- 6) K. Okuyama, M. Takayasu and H. Takayasu, *Physica A* **269** (1999), 125.
R. L. Axtell, *Science* **293** (2001), 1818.