

## Yang-Mills Gravity and Gravitational Radiation

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A new translation gauge theory of gravity in flat space-time with a new Yang-Mills action involves quadratic gauge curvature and fermions. The theory shows that the presence of an “effective Riemannian metric tensor” for the motions of classical particles and light rays is probably the manifestation of the translation gauge symmetry in flat physical space-time. In the post-Newtonian approximation of the tensor gauge field produced by the energy-momentum tensor, the results are shown to be consistent with classical tests of gravity and with the quadrupole radiations of binary pulsars.

To explore quantization of gravity, we consider a spacetime-translation gauge theory of gravity with a new Yang-Mills action involves quadratic gauge curvature in flat spacetime. An “effective Riemannian metric tensor” emerges only in the limit of geometric optics (or classical limit) of wave equations of matter and radiation, and describes only the motions of classical particles and light rays. The presence of this effective metric tensor appears to be a manifestation of the translation gauge symmetry in flat physical space-time. From the viewpoint of Yang-Mills gravity, ‘metric tensors’ exist only in the classical limit and are not fundamental fields. This result appears to have advantages for accommodating both quantum mechanics and gravity.<sup>1)</sup> The Yang-Mills gravity is shown to be consistent with quadrupole radiations and with classical tests of gravity. A preliminary study shows that the gravity based on such a Yang-Mills action in flat spacetime can be quantized and is less divergent than that of general relativity.

We generalize the usual Yang-Mills theory with internal gauge groups to a gauge theory with the external space-time translation group in which the generators of the group do not have constant matrix representations. This generalization appears to be essential for the existence of the tensor gauge fields which are generated by the well-defined and conserved energy-momentum tensor, in analogy with electrodynamics in which the electromagnetic field is generated by the conserved charge. The tensor nature of this gauge field dictates that there is only one kind of force between all matter and anti-matter. Furthermore, the translation gauge symmetry implies that the coupling constant  $g$  of the tensor gauge field has the dimension of length ( $c=\hbar=1$ ), in contrast to the dimensionless  $g_{YM}$  in usual Yang-Mills theories.

The Lagrangian formulations of Yang-Mills (and electromagnetic) theories associated with internal gauge groups are all based on the replacement,  $\partial_\mu \rightarrow \partial_\mu + ig_{YM} B_\mu$ , ( $B_\mu = B_\mu^a t_a$ ), which involves a dimensionless coupling constant  $g_{YM}$  and dimensionless constant matrix representations of the generators  $t_a$  of the gauge groups

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in inertial frames. However, the generators of the external space-time translation group  $T(4)$  are the displacement operators,  $p^\mu = i\partial^\mu$ ; thus, the replacement takes a non-Yang-Mills form,

$$\partial^\mu \rightarrow \partial^\mu - ig\phi^{\mu\nu}p_\nu \equiv J^{\mu\nu}\partial_\nu, \quad J^{\mu\nu} = \eta^{\mu\nu} + g\phi^{\mu\nu}, \quad (1)$$

where  $\eta^{\mu\nu} = (1, -1, -1, -1)$ . In contrast to the usual Yang-Mills theories, the generators of this external translation group  $T(4)$  is  $p^\mu = i\partial^\mu$ , we have a symmetric tensor gauge field  $\phi^{\mu\nu} = \phi^{\nu\mu}$  (i.e., a spin-2 field) rather than a 4-vector field (i.e., a spin-1 field) in the gauge covariant derivative,  $\Delta^\mu = J^{\mu\nu}\partial_\nu$  (in inertial frames).

In order to understand both gravity and accelerated expansion of baryon matter in the universe, one can construct a theory based on the gauge group  $T(4) \times U(1)$ .\*) Here, we concentrate on the external gauge group of space-time translations  $T(4)$ , which is the Abelian subgroup of the Poincaré group and is non-compact. This group  $T(4)$  is particularly interesting because it is the minimal group related to the conserved energy-momentum tensor, which couples to a tensor (or spin-2) field  $\phi_{\mu\nu}$  and is the source of the gravitational field.<sup>2)</sup>

The translation gauge symmetry is based on the local space-time translation with an arbitrary infinitesimal vector gauge-function  $A^\mu(x)$ ,

$$x^\mu \rightarrow x'^\mu = x^\mu + A^\mu(x), \quad x \equiv x^\lambda = (w, x, y, z). \quad (2)$$

This transformation has a dual interpretation: (i) a shift of the space-time coordinates by an infinitesimal vector gauge-function  $A^\mu(x)$ , and (ii) an arbitrary infinitesimal transformation. For the interpretation (ii) to be consistent in the theory, we must formulate Yang-Mills gravity for both inertial and non-inertial frames (i.e., general frames). For simplicity, we can accommodate these two mathematical implications of the transformation (2) by defining a gauge transformation of space-time translations for physical quantities such as scalar, vector and tensor fields are\*\*)

$$Q(x) \rightarrow (Q(x))^\S = Q(x) - \Lambda^\lambda \partial_\lambda Q(x), \quad Q(x) = \psi, \bar{\psi}, \Phi, \quad (3)$$

$$D_\mu Q \rightarrow (D_\mu Q)^\S = D_\mu Q - \Lambda^\lambda \partial_\lambda (D_\mu Q) - (D_\lambda Q) \partial_\mu \Lambda^\lambda, \quad (4)$$

$$\Gamma^\mu \rightarrow (\Gamma^\mu)^\S = \Gamma^\mu - \Lambda^\lambda \partial_\lambda \Gamma^\mu + \Gamma^\lambda \partial_\lambda \Lambda^\mu, \quad (5)$$

$$T_{\mu\nu} \rightarrow (T_{\mu\nu})^\S = T_{\mu\nu} - \Lambda^\lambda \partial_\lambda T_{\mu\nu} - T_{\mu\alpha} \partial_\nu \Lambda^\alpha - T_{\alpha\nu} \partial_\mu \Lambda^\alpha, \quad T_{\mu\nu} = J_{\mu\nu}, P_{\mu\nu}, \quad (6)$$

$$Q^{\mu\nu} \rightarrow (Q^{\mu\nu})^\S = Q^{\mu\nu} - \Lambda^\lambda \partial_\lambda Q^{\mu\nu} + Q^{\lambda\nu} \partial_\lambda \Lambda^\mu + Q^{\mu\lambda} \partial_\lambda \Lambda^\nu. \quad (7)$$

\*) The  $U(1)$  gauge field here is associated with the established baryon number conservation, as suggested by T. D. Lee and C. N. Yang (Phys. Rev. **98** (1955), 1504). If the gauge invariant Lagrangian takes a modified form  $\partial_\lambda F_{\mu\nu} \partial^\lambda F^{\mu\nu}$ , one has a fourth-order field equation and a static linear potential, which can produce a long range repulsive force between observable baryon matter in the universe.

\*\*) The general gauge transformation for  $Q_{\alpha_1 \dots \alpha_n}^{\mu_1 \dots \mu_m}(x)$  is:  $Q_{\alpha_1 \dots \alpha_n}^{\mu_1 \dots \mu_m}(x) \rightarrow$

$$(Q_{\alpha_1 \dots \alpha_n}^{\mu_1 \dots \mu_m}(x))^\S = \left( Q_{\beta_1 \dots \beta_n}^{\nu_1 \dots \nu_m}(x) - \Lambda^\lambda(x) \partial_\lambda Q_{\beta_1 \dots \beta_n}^{\nu_1 \dots \nu_m}(x) \right) \frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}} \dots \frac{\partial x'^{\mu_m}}{\partial x^{\nu_m}} \frac{\partial x^{\beta_1}}{\partial x'^{\alpha_1}} \dots \frac{\partial x^{\beta_n}}{\partial x'^{\alpha_n}},$$

where  $\mu_1, \nu_1, \alpha_1, \beta_1$ , etc. are space-time indices. See Ref. 2).

Here  $D_\mu$  denotes the partial covariant derivative associated with a metric tensor  $P_{\mu\nu}(x)$  in a general reference frame (inertial or non-inertial<sup>2)</sup>). Note that the functions  $D_\mu Q$  and  $D_\mu D_\nu Q$  transform, by definition, as a covariant vector  $Q_\mu(x)$  and a covariant tensor  $Q_{\mu\nu}(x)$  respectively under the translational gauge transformation. As usual, both (Lorentz) spinor field  $\psi$  and (Lorentz) scalar field  $\Phi$  are treated as ‘coordinate scalars’ and have the same translational gauge transformation:

$$\psi \rightarrow \psi^{\mathfrak{s}} = \psi - \Lambda^\lambda \partial_\lambda \psi, \quad \Phi \rightarrow \Phi^{\mathfrak{s}} = \Phi - \Lambda^\lambda \partial_\lambda \Phi. \quad (8)$$

To see the field-theoretic origin of an effective Riemannian metric tensors, let us consider the kinetic term for a scalar field  $\phi$  in a Lagrangian in an inertial frame ( $P_{\mu\nu} = \eta_{\mu\nu}, D_\mu = \partial_\mu$ ) for simplicity. For the presence of the gauge field  $\phi_{\mu\nu}$ , the translation gauge symmetry dictates the replacement (1). We obtain

$$(1/2)\eta_{\mu\nu}(\partial^\mu \Phi)(\partial^\nu \Phi) \rightarrow (1/2)G^{\mu\nu}(\partial_\mu \Phi)(\partial_\nu \Phi), \quad (9)$$

$$G^{\mu\nu} = \eta^{\mu\nu} + 2g\phi^{\mu\nu} + g^2\phi^{\mu\alpha}\phi^{\nu\beta}\eta_{\alpha\beta}. \quad (10)$$

In the literature, when one arrives at this crucial step (9),<sup>3)</sup> one usually gives up the Yang-Mills approach for a truly gauge invariant theory with a quadratic gauge curvature in a flat space-time, and follows Einstein’s approach to discuss gravity by postulating Riemannian space-time due to the presence of  $\phi_{\mu\nu}$  in (10).

We stress that, from the viewpoint of Yang-Mills theory, the real physical space-time in (9) is still flat and the fundamental metric tensor is still  $P_{\mu\nu}$  in general frames of reference. We shall take this viewpoint throughout the discussion.

In the generalized Yang-Mills theory with the external translation gauge symmetry, we have the gauge curvature

$$C^{\mu\nu\alpha} = J^{\mu\lambda}(D_\lambda J^{\nu\alpha}) - J^{\nu\lambda}(D_\lambda J^{\mu\alpha}), \quad (11)$$

which is given by the commutation relation of the gauge covariant derivative  $\Delta^\mu = J^{\mu\nu}D_\nu$ , i.e.,  $[\Delta^\mu, \Delta^\nu] = C^{\mu\nu\alpha}D_\alpha$ . We postulate that, in a general frame, the action  $S_{\phi\psi}$  for fermion matter and spin-2 fields involves the linear combination of the two independent quadratic terms of the gauge-curvature and the fermion Lagrangian:

$$S_{\phi\psi} = \int L_{\phi\psi} \sqrt{-P} d^4x, \quad P = \det P_{\mu\nu}, \quad (12)$$

$$L_{\phi\psi} = \frac{1}{2g^2} \left( C_{\mu\alpha\beta} C^{\mu\beta\alpha} - C_{\mu\alpha}{}^\alpha C^{\mu\beta}{}_\beta \right) + \frac{i}{2} [\bar{\psi} \Gamma_\mu \Delta^\mu \psi - (\Delta^\mu \bar{\psi}) \Gamma_\mu \psi] - m \bar{\psi} \psi, \quad (13)$$

$$\Delta^\mu \psi = J^{\mu\nu} D_\nu \psi, \quad J^{\mu\nu} = P^{\mu\nu} + g\phi^{\mu\nu} = J^{\nu\mu}, \quad D_\lambda P_{\mu\nu} = 0. \quad (14)$$

As usual, we include a suitable gauge-fixing term in the Lagrangian to make the solutions of gauge field equation well-defined. The Lagrange equations for the gravitational tensor field  $\phi^{\mu\nu}$  in (1) can be derived from the action  $\int L_{tot} d^4x$ , where  $L_{tot}$  is the original Lagrangian  $L_{\phi\psi}$  with an additional gauge-fixing term<sup>2)</sup> involving the gauge parameter  $\xi$ . We obtain

$$H^{\mu\nu} + \xi A^{\mu\nu} = g^2 T^{\mu\nu}, \quad (15)$$

$$\begin{aligned}
H^{\mu\nu} \equiv & \partial_\lambda (J_\rho^\lambda C^{\rho\mu\nu} - J_\alpha^\lambda C^{\alpha\beta}{}_\beta \eta^{\mu\nu} + C^{\mu\beta}{}_\beta J^{\nu\lambda}) \\
& - C^{\mu\alpha\beta} \partial^\nu J_{\alpha\beta} + C^{\mu\beta}{}_\beta \partial^\nu J_\alpha^\beta - C^{\lambda\beta}{}_\beta \partial^\nu
\end{aligned} \tag{16}$$

$$A^{\mu\nu} = \partial^\mu \left( \partial^\lambda J_\lambda^\nu - \frac{1}{2} \partial^\nu J_\lambda^\lambda \right) - \frac{1}{2} \eta^{\mu\nu} \left( \partial^\alpha \partial^\lambda J_{\lambda\alpha} - \frac{1}{2} \partial^\alpha \partial_\alpha J_\lambda^\lambda \right), \tag{17}$$

where  $\mu$  and  $\nu$  should be made symmetric. The energy-momentum tensor  $T^{\mu\nu}$  of fermion matter is given by  $T^{\mu\nu} = \frac{1}{2} [\bar{\psi} i \Gamma^\mu \partial^\nu \psi - i (\partial^\nu \bar{\psi}) \Gamma^\mu \psi]$ .

In Yang-Mills gravity, the gravitational quadrupole radiations of binary pulsars can be calculated to the second-order in  $g\phi^{\mu\nu}$  in inertial frames. The energy-momentum tensor  $t_{\mu\nu}$  of gravitation is defined by the exact field equation (15) (with  $\xi = 0$ ) written in the following form,

$$\partial^\lambda \partial_\lambda \phi^{\mu\nu} = -g(T^{\mu\nu} + t^{\mu\nu}), \tag{18}$$

$$\begin{aligned}
t^{\mu\nu} = & \frac{1}{g^2} [C^{\rho\mu\nu} \partial_\lambda J_\rho^\lambda + \partial_\rho (g\phi^{\rho\lambda} \partial_\lambda J^{\mu\nu}) + g\phi_\rho^\lambda \partial_\lambda (J^{\rho\alpha} \partial_\alpha J^{\mu\nu}) - J^\lambda{}_\rho \partial_\lambda (J^{\mu\alpha} \partial_\alpha J^{\rho\nu}) \\
& - C^{\mu\beta\alpha} \partial^\nu J_{\alpha\beta} - P^{\mu\nu} \partial_\lambda (J^\lambda{}_\rho C^{\rho\beta}{}_\beta) + \partial_\lambda (C^{\mu\beta}{}_\beta J^{\nu\lambda}) + C^{\mu\beta}{}_\beta \partial^\nu J^\lambda{}_\lambda - C^{\lambda\beta}{}_\beta \partial^\nu J_\lambda^\mu]. \tag{19}
\end{aligned}$$

It suffices to take a second order approximation in  $g\phi_{\mu\nu}$  for our calculations. With the help of the gauge condition  $\partial_\mu \phi^{\mu\nu} = \partial^\nu \phi_\lambda^\lambda / 2$ , the result for the energy-momentum tensor of the gravitational field is still complicated. However, we can simplify it by taking the average of  $t^{\mu\nu}$  over a region of space and time much larger than the wavelengths of the radiated waves. Following the usual method,<sup>4)</sup> the total power  $P_o$  emitted by a rotating body (rotating around one of the principal axes of the ellipsoid of inertia) at twice the rotating frequency (i.e.,  $\omega = 2\Omega$ ) is given by

$$P_o(\omega) = 32G\Omega^6 I^2 e^2 / 5, \tag{20}$$

where  $I$  is the moment of inertia and  $e$  is the equatorial ellipticity. To the second-order approximation, the result (20) for the power emitted per solid angle in Yang-Mills gravity turns out to be the same as that obtained in general relativity and consistent with the data of the binary pulsar PSR 1913+16.<sup>4)</sup> Furthermore, Yang-Mills gravity can be verified to be consistent with other experiments.<sup>5),\*)</sup>

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\*) By the usual successive approximations, the advance of the perihelion for one revolution of the planet and the bending of light are given by

$$\delta\phi = \frac{6\pi Gm}{P} \left( 1 - \frac{3(E_o^2 - m_p^2)}{4m_p^2} \right), \quad \text{and} \quad \Delta\phi \approx \frac{4Gm\omega_o}{M} \left( 1 - \frac{18G^2 m^2 \omega_o^2}{M^2} \right).$$

We note that the second terms in the brackets of these two equations show the differences between the present Yang-Mills gravity and Einstein's theory. However, these differences are too small to be detected by the known experiments inside and outside the solar system. For example, the observational accuracy of the perihelion shift of the Mercury is about 1 %, the difference between Yang-Mills gravity and Einstein gravity can be tested only if the Mercury were to move with a tenth of the speed of light.

## References

- 1) F. J. Dyson, in *100 Years of Gravity and Accelerated Frames—The Deepest Insights of Einstein and Yang-Mills*, ed. J. P. Hsu and D. Fine (World Scientific, 2005), p. 348.
- 2) J. P. Hsu, *Int. J. Mod. Phys. A* **21** (2006), 5119.  
See also J. P. Hsu, in *100 Years of Gravity and Accelerated Frames, The Deepest Insights of Einstein and Yang-Mills* ed. J. P. Hsu and D. Fine (World Scientific, 2005), p. 462.
- 3) R. Utiyama and T. Fukuyama, *Prog. Theor. Phys.* **45** (1971), 612.  
Y. M. Cho, *Phys. Rev. D*, **14** (1976), 2521; *Phys. Rev. D* **14** (1976), 3341.
- 4) S. Weinberg, *Gravitation and Cosmology* (John Wiley and Sons, 1972), p. 7, p. 175 and p. 251.  
J. H. Taylor, *Rev. Mod. Phys.* **66** (1994), 711.  
T. Damour and J. H. Taylor, *Phys. Rev. D* **45** (1992), 1840.
- 5) L. Landau and E. Lifshitz, *The Classical Theory of Fields*, trans. by M. Hamermesh (Addison-Wesley, Cambridge, Mass., 1951), p. 276 and p. 312.